



Numerical Solution of Volterra-Hammerstein Integral Equation of the First Kind by Finite Difference Method Decomposition with Nyström Method

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ABSTRACT

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According to common knowledge, the first kind of Volterra integral equation is an example of a problem that is not well-posed. This kind of equation is encountered in several problems of science, and it is useful in a variety of fields, including control theory, nuclear reactors, and ecological systems, where it can be used to evolutionary processes. In this article, we'll present an effective and accurate technique to converting VK1 into those of the second kind (VK2) and the kernel must not be zero for the conversion process to be effective, and we will find the approximation solutions of them by using the decomposition of Taylor series with Nyström method (Trapezoidal and Simpson's rules). In finality, we will present a variety of numerical examples to demonstrate that the conversion that has been proposed is both successful and stable.

1. INTRODUCTION

In this work, a direct regularization technique for solving nonlinear ill-posed operator equations will be presented:

$$A\phi = f \quad (1)$$

where the nonlinear operator $A: D \rightarrow F$ is defined for $\phi \in D$ and $f \in F$ which are Hilbert or Banach spaces.

It is well known that the first kind Volterra integral equation (VK1) is inherently ill-posed. In literature, Tikhonov [1] and Phillips [2] developed the regularization techniques for these ill-posed equations, Eq. (1). Many different methods of regularizing VK1 have been presented like as Kabanikhin method [3] and Denisov method [4]. For further information and additional details on regularization and computational solutions for ill-posed Volterra equations, refer to references [1-3, 5-12].

The theory and applications of integral equations such as elasticity, semi-conductors, scattering theory, metallurgy, seismology, thermal conditions, fluid flow, population dynamics, chemical processes, etc., are discussed in the book "Leçons sur les équations intégrales et intégrales-différentielles" written by Vito Volterra. This book was published in 1913, illustrated by Wazwaz [5].

Concerning the subject of this work is the first kind nonlinear Volterra integral equation of the form:

$$\int_a^t k(t,x)F(\phi(x))dx = f(t), \quad a \leq x \leq t \leq b \quad (2)$$

where $\phi(t)$ is unknown function, $F(\phi(x))$ is a nonlinear

function of $\phi(x)$, $k(t,x)$ is a kernel and $f(t)$ is given function in $a \leq x \leq t \leq b$ with $f(a) = 0$, we will take:

$$A\phi(t) : A(t) = \int_a^t k(t,x)F(\phi(x))dx. \quad t \in [a,b]$$

2. MAIN RESULT

In this part, we'll transmit the nonlinear integral equation Eq. (2) to a second kind integral equation defined in the interval $I=[0,1]$. It is necessary to highlight that the solution for ill-posed problems is generally unstable, and slight changes can make large errors. The Leibnitz theorem and the Taylor series are going to be applied in the next section:

Let $A(t)$ be a function with n derivatives with respect to t in $I=[0,1]$ than for $0 < t - \varepsilon < t < t + \varepsilon < 1$, with $\varepsilon \rightarrow 0$. The *Taylor series* is given by

$$\begin{aligned} A(t + \varepsilon) &= A(t) + \frac{\varepsilon}{1!} \cdot \frac{\partial A(t)}{\partial t} + \frac{\varepsilon^2}{2!} \cdot \frac{\partial^2 A(t)}{\partial t^2} + \dots \\ &\quad + \frac{\varepsilon^n}{n!} \cdot \frac{\partial^n A(t)}{\partial t^n} + O(\varepsilon^n), \\ A(t - \varepsilon) &= A(t) - \frac{\varepsilon}{1!} \cdot \frac{\partial A(t)}{\partial t} + \frac{\varepsilon^2}{2!} \cdot \frac{\partial^2 A(t)}{\partial t^2} - \dots \\ &\quad + (-1)^n \frac{\varepsilon^n}{n!} \cdot \frac{\partial^n A(t)}{\partial t^n} + O(\varepsilon^n), \end{aligned}$$

where, $O(\varepsilon^n)$ is the approximation error term. For the

derivation of higher order approximations to derivatives of any order, the Taylor expansion is a very helpful tool. If ε is small, then higher order accuracy generally means higher accuracy. The first order of Taylor series is given by:

$$A(t + \varepsilon) = A(t) + \frac{\varepsilon}{1!} \frac{\partial A(t)}{\partial t} + O(\varepsilon) \quad (3)$$

$$A(t - \varepsilon) = A(t) - \frac{\varepsilon}{1!} \frac{\partial A(t)}{\partial t} + O(\varepsilon) \quad (4)$$

The Leibnitz rule is a famous rule utilizing for differentiation of integrals [5]. Let $A(t)$ and $\frac{\partial A(t)}{\partial t}$ be continuous in the domain $0 \leq t \leq 1$, and let:

$$A(t) = \int_{g(t)}^{h(t)} H(t, x) dx,$$

then differentiation of the integral in $A(t)$ exists and is given by:

$$\begin{aligned} \frac{\partial A(t)}{\partial t} &= H(t, h(t)) \frac{dh(t)}{dt} - H(t, g(t)) \frac{dg(t)}{dt} \\ &+ \int_{g(t)}^{h(t)} \frac{\partial H(t, x)}{\partial t} dx. \end{aligned}$$

In our previous work [12, 13], we converted the Volterra-Hammerstein integral Eq. (2) to a second kind integral equation defined in the interval $[0, 1]$ by using Taylor series of the first order Eq. (3) (This approximation is known as: "fda" the forward difference approximate of $A(t)$), we obtained this equivalent equation:

$$\begin{aligned} \int_a^{t+\varepsilon} k(t+\varepsilon, x) F(\phi_\varepsilon(x)) dx &= f(t) + \varepsilon k(t, t) F(\phi_\varepsilon(t)) \\ + \varepsilon \int_a^t \frac{\partial k}{\partial t}(t, x) F(\phi_\varepsilon(x)) dx &= f(t + \varepsilon), \end{aligned}$$

then, we get a well-posed integral equation (VK2), which is given by:

$$F(\phi_\varepsilon(t)) + \int_a^t \frac{k(t, x) + \varepsilon \frac{\partial k(t, x)}{\partial t}}{\varepsilon k(t, t)} F(\phi_\varepsilon(x)) dx = \frac{f(t + \varepsilon)}{\varepsilon k(t, t)}, \quad (5)$$

$\varepsilon \rightarrow 0$, and $k(t, t) \neq 0$.

Eq. (5) can be rewritten as follows:

$$\Phi_\varepsilon(t) + \int_0^t K(t, x) \Phi_\varepsilon(x) dx = f_\varepsilon(t) \quad (6)$$

where,

$$K(t, x) = \frac{k(t, x) + \varepsilon \frac{\partial k(t, x)}{\partial t}}{\varepsilon k(t, t)}, \quad f_\varepsilon(t) = \frac{f(t + \varepsilon)}{\varepsilon k(t, t)}$$

$$\Phi_\varepsilon = F(\phi_\varepsilon(t)) = \Phi \text{ if } \varepsilon \rightarrow 0.$$

and $\phi_\varepsilon(t) = F^{-1}(\Phi_\varepsilon(t)) = \phi(t)$ if $\varepsilon \rightarrow 0$. Substituting $t = 0$ into Eq. (6) gives the initial condition $\Phi_\varepsilon(0) = \Phi_0$. For more details, refer to references [11-13].

If we use the Taylor series of the first order Eq. (4) (the first order backward difference approximation "bda"), and Leibnitz rule, we get a well-posed integral equation (VK2), which is given by:

$$\begin{aligned} F(\phi_\varepsilon(t)) + \int_a^t \frac{-k(t, x) + \varepsilon \frac{\partial k(t, x)}{\partial t}}{\varepsilon k(t, t)} F(\phi_\varepsilon(x)) dx \\ = \frac{-f(t - \varepsilon)}{\varepsilon k(t, t)}, \quad \varepsilon \rightarrow 0, \text{ and } k(t, t) \neq 0. \end{aligned} \quad (7)$$

Eq. (7) can be simplified as follows:

$$\Phi_\varepsilon(t) + \int_0^t K(t, x) \Phi_\varepsilon(x) dx = f_\varepsilon(t),$$

where

$$K(t, x) = \frac{k(t, x) + \varepsilon \frac{\partial k(t, x)}{\partial t}}{\varepsilon k(t, t)}$$

$$f_\varepsilon(t) = \frac{-f(t - \varepsilon)}{\varepsilon k(t, t)}$$

$$\Phi_\varepsilon = F(\phi_\varepsilon(t)) = \Phi \text{ if } \varepsilon \rightarrow 0$$

Now, if we use Eq. (3) and Eq. (4) (the central difference approximation "cda") of the flowing form:

$$A(t + \varepsilon) - A(t - \varepsilon) \approx 2 \frac{\varepsilon}{1!} A'(t) + O(\varepsilon), \quad \varepsilon \rightarrow 0$$

and Leibnitz rule, we get a well-posed integral equation (VK2), which is given by:

$$\begin{aligned} F(\phi_\varepsilon(t)) + \int_a^t \frac{k'(t, x)}{k(t, t)} F(\phi_\varepsilon(x)) dx \\ = \frac{f(t + \varepsilon) - f(t - \varepsilon)}{2\varepsilon k(t, t)}, \quad \varepsilon \rightarrow 0, \text{ and } k(t, t) \neq 0. \end{aligned} \quad (8)$$

Eq. (8) can be rewritten in the form:

$$\Phi_\varepsilon(t) + \int_0^t K(t, x) \Phi_\varepsilon(x) dx = f_\varepsilon(t),$$

where,

$$K(t, x) = \frac{k'(t, x)}{k(t, t)}, \quad f_\varepsilon(t) = \frac{f(t + \varepsilon) - f(t - \varepsilon)}{2\varepsilon k(t, t)}$$

$$\Phi_\varepsilon = F(\phi_\varepsilon(t)) = \Phi \text{ if } \varepsilon \rightarrow 0.$$

Now, we state the theorem of the existence and uniqueness of the solution to Volterra integral equation of the first kind [14, 15]:

Theorem [14]: Assume that,

- 1) $k(t, x)$ and $\frac{\partial k(t, x)}{\partial t}$ are continuous in $0 \leq x \leq t \leq T$,
- 2) $k(t, t)$ does not vanish anywhere in $0 \leq x \leq t \leq T$,
- 3) $f(0) = 0$,
- 4) $f(t)$ and $f'(t)$ are continuous in $0 \leq x \leq t \leq T$.

Then:

$$\int_0^t k(t, x)\Phi(x)dx = f(t), \quad 0 \leq x \leq t \leq T$$

has a unique continuous solution. This solution is identical with the continuous solution of

$$k(t, x)\Phi(x) + \int_a^t \frac{\partial k(t, x)}{\partial t} \Phi(x)dx = f'(t).$$

There are different iterative methods available for resolving nonlinear Volterra integral equations such as variational iteration method, Adomian's decomposition method, and homotopy perturbation method [16-18]. In our previous research [12], we applied the variational iteration method with Taylor series for solving the above ill-posed problem Eq. (2) which is equivalent to the well-posed problems Eq. (5), Eq. (7) and Eq. (8). Having converted the above integral equation of the first kind to the linear Volterra integral equation of the second kind, we then can use any numerical method like Nyström methods; trapezoidal method, Simpson method, modified Simpson method.

In this section, we shall describe the quadrature or Nyström method (Trapezoidal rule and Simpson's rule...), for the approximate solution of linear Volterra integral equations of the second kind with continuous kernels.

The quadrature methods are intended to estimate the definite integral of $f(t)$ over the interval $I = [a, b]$ by evaluating $f(t)$ at a finite number of sample points.

Assume that:

$$a = t_1^{(n)} < t_2^{(n)} < \dots < t_n^{(n)} = b.$$

A form of formula:

$$Q_n[f] = \sum_{i=1}^n w_i^{(n)} f(t_i^{(n)}),$$

with a property that

$$\int_a^b f(t)dt = Q_n[f] + E[f],$$

The term $E[f]$ is called the truncation error for integration. The values $\{t_i^{(n)}\}_{i=1}^n$ are called the quadrature nodes and $\{w_i^{(n)}\}_{i=1}^n$ are called the weights. A sequence $Q[f]$ of quadrature formulas is called convergent if $Q_n[f] \rightarrow Q[f]$,

$n \rightarrow \infty$, for all $f \in C(I)$ [16].

Corollary [19] (Trapezoidal rule: Error analysis)

Suppose that $[a, b]$ is subdivided into n subintervals $[t_i, t_{i+1}]$ of width $h = (b - a)/n$. The composite trapezoidal rule:

$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h \sum_{i=1}^{n-1} f(x_i)$$

is an approximation to the integral

$$\int_a^b f(x)dx = T(f, h) + E_T(f, h).$$

Furthermore, if $f \in C^2[a, b]$, there exists a value c with $a < c < b$ so that the error term $E_T(f, h)$ has the form:

$$E_T(f, h) = \frac{-(b-a)f^{(2)}(c)h^2}{12} = O(h^2).$$

The linear integral of the second kind can be mathematically approximated using any quadrature rule, as shown below:

$$\int_I k(t, x)\phi(x)dx \approx \sum_{j=1}^n w_j k(t, x_j)\phi(x_j),$$

with quadrature points (nodes) $\{x_j\}_{j=1}^n$ contained in I and real quadrature weights $\{w_j\}_{j=1}^n$.

$$\phi_n(t) + \sum_{j=1}^n w_j k(t, x_j)\phi_n(x_j) = f(t),$$

where $\phi_n(t)$ is an approximation to $\phi(t)$.

By the numerical integration formulas of Trapezium rule, so we get

$$\phi(t_j) = f(t_j) + \frac{h}{2} \left[k(t_j, t_1)\phi(t_1) + 2 \sum_{i=2}^{j-1} k(t_j, t_i)\phi(t_i) + k(t_j, t_j)\phi(t_j) \right]$$

We will now apply modified Simpson method and take $\Phi_\epsilon(t) = \Phi(t)$. Consider let

$$t_0 = 0 < t_1 < \dots < t_{2j} < \dots < t_{2n} = 1,$$

be a step's equidistant subdivision $h = t_{2j+1} - t_{2j}$ for $j = 0, 1, \dots, n$. The goal is to approximate the solutions of the approximation of the second kind equation Eq. (5), Eq. (7) or Eq. (8) to the all nodes of $2j$ indices (at the point t_{2j}), then the form of modified Simpson method is:

$$\int_{t_{2j}}^{t_{2j+2}} g(t)dt = \frac{h}{3} [g(t_{2j}) + 4g(t_{2j+1}) + g(t_{2j+2})],$$

where the integration error is $O(h) = -2 \frac{(h/2)^5}{90} (g(\zeta))^{(4)}$.

This method has been used by Nadir and Rahmoune [20].

By using this method can be written the equation Eq. (5), Eq. (7) and Eq. (8) of the second kind in the algorithm of the following form:

$$\Phi(t_{2j}) = \frac{h}{3} \sum_{i=0}^{j-1} \left[K(t_{2j}, x_{2i}) \Phi(t_{2i}) + 4K(t_{2j}, x_{2i+1}) \Phi(t_{2i+1}) + K(t_{2j}, x_{2i+2}) \Phi(t_{2i+2}) \right] + f(t_{2j}),$$

We approximate Φ_{2i+1} by $\frac{\Phi_{2i} + \Phi_{2i+2}}{2}$, the Eq. (5) becomes:

$$\begin{aligned} \Phi_{2j} & \left(1 - \frac{h}{3} (2K_{2j,2j-1} + K_{2j,2j}) \right) \\ & = f_{2j} + \left(\frac{h}{3} (K_{2j,0} + 2K_{2j,1}) \right) \Phi_0 \\ & + \frac{2h^2}{3} \sum_{i=0}^{j-1} (K_{2j,2i} + 2K_{2j,2i+1}) \Phi_{2i} \\ & + \frac{2h^2}{3} \sum_{i=0}^{j-1} (2K_{2j,2i+1} + K_{2j,2i+2}) \Phi_{2i+2}. \end{aligned}$$

We are able to calculate the approximate solutions Φ of the equations Eq. (5), Eq. (7) and Eq. (8) by using recurrence. in all points t_{2j} for $j = 0, 1, \dots, n$. It is evident that the initial value of Φ is $\Phi(0) = \Phi_0 = f_\varepsilon(0)$. Suppose that $F(\phi_\varepsilon(t))$ is invertible. After that, we will be able to set:

$$\phi_\varepsilon = F^{-1}(\Phi(t))$$

3. NUMERICAL EXAMPLES

Our method of conversion for nonlinear ill-posed Volterra equations will be demonstrated by discussing examples that follow, and we will compare the numerical results between three approximate Eqs. (5)-(8). Rather than conducting an exhaustive investigation of the numerical properties, the objective is to illustrate the viability of the proposed method.

Example 1: [12]

Let be a Volterra -Hammerstein integral equation of the form:

$$\int_0^t (10t + 6 - 10x) \log |\phi(x)| dx = 9t^2 + 5t^3, \quad 0 \leq x \leq t \leq 1$$

To solve this equation, first we convert it to linear VK2 of the second kind (VK2_{fda}, VK2_{bda} and VK2_{cda}, respectively) given by:

$$\Phi_\varepsilon(t) + \int_0^t \frac{10t - 10x + 6 + 10\varepsilon}{6\varepsilon} \Phi_\varepsilon(x) dx = \frac{9(t + \varepsilon)^2 + 5(t + \varepsilon)^3}{6\varepsilon}$$

$$\Phi_\varepsilon(t) + \int_0^t \frac{-10t + 10x - 6 + 10\varepsilon}{6\varepsilon} \Phi_\varepsilon(x) dx = \frac{9(t - \varepsilon)^2 + 5(t - \varepsilon)^3}{-6\varepsilon}$$

$$\Phi_\varepsilon(t) + \int_0^t \frac{5}{3} \Phi_\varepsilon(x) dx = \frac{9(t + \varepsilon)^2 + 5(t + \varepsilon)^3 - 9(t - \varepsilon)^2 - 5(t - \varepsilon)^3}{12\varepsilon}$$

for $t \in [0, 1]$ with $\Phi_\varepsilon(t) = \log |\phi(t)|$, the exact solution is:

$$\phi(t) = e^{3t}.$$

Table 1. Comparison of the absolute errors for Example 1 of VK2_{fda}, VK2_{bda} and VK2_{cda} obtained by Taylor approximation ($\varepsilon = 10^{-4}$) and Nyström method ($n = 20$)

t_{2j} Simpson	Error of VK2 _{fda}	Error of VK2 _{cda}	Error of VK2 _{bda}
0	1.5002e-04	8.3333e-09	1.4998e-04
0.1	2.0080e-04	9.5217e-09	2.2815e-04
0.2	2.6877e-04	1.0879e-08	3.4707e-04
0.3	3.5974e-04	1.2431e-08	5.2797e-04
0.4	4.8151e-04	1.4204e-08	8.0316e-04
0.5	6.4450e-04	1.6229e-08	1.2218e-03
0.6	8.6265e-04	1.8541e-08	1.8586e-03
0.7	1.1547e-03	2.1188e-08	2.8274e-03
0.8	1.5455e-03	2.4189e-08	4.3011e-03
0.9	2.0687e-03	2.7648e-08	6.5430e-03
1	2.7689e-03	3.1610e-08	9.9534e-03
t_{2j} Trapezoidal	Error of VK2 _{fda}	Error of VK2 _{cda}	Error of VK2 _{bda}
0	1.5002e-04	8.3333e-09	1.4998e-04
0.1	6.4960e-04	9.5210e-09	1.2329e-03
0.2	2.0299e-03	1.0878e-08	3.1277e-03
0.3	4.3391e-03	1.2428e-08	6.3570e-03
0.4	8.1256e-03	1.4199e-08	1.1770e-02
0.5	1.4252e-02	1.6223e-08	2.0743e-02
0.6	2.4076e-02	1.8533e-08	3.5511e-02
0.7	3.9728e-02	2.1177e-08	5.9687e-02
0.8	6.4551e-02	2.4176e-08	9.9128e-02
0.9	1.0379e-01	2.7631e-08	1.6330e-01
1	1.6569e-01	3.1588e-08	2.6752e-01

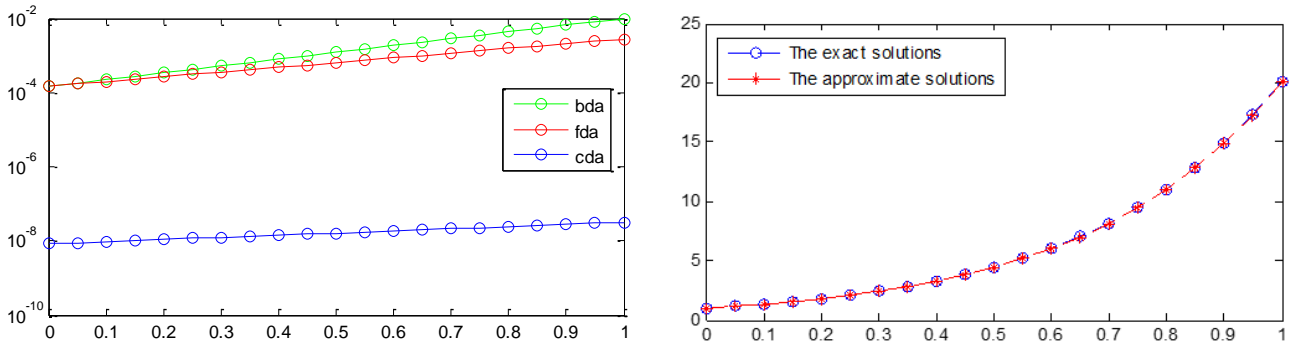


Figure 1. Comparison of results to Example 1, for $n=20$ and $\varepsilon = 10^{-4}$ by Simpson method

Example 2: [8]

Let be a Volterra -Hammerstein integral equation of the form:

$$\int_0^t (\sin(t-x)+1)\cos(\phi(x))dx = \frac{t \sin t}{2} + \sin t,$$

$$0 \leq x \leq t \leq 1$$

To solve this equation, first we convert it to linear VK2 of the second kind (VK2_{fda}, VK2_{bda} and VK2_{cda}, respectively) given by:

$$\Phi_\varepsilon(t) + \int_0^t \frac{\sin(t-x)+1+\varepsilon \cos(t-x)}{\varepsilon} \Phi_\varepsilon(x)dx$$

$$= \frac{(t+\varepsilon+2)\sin(t+\varepsilon)}{2\varepsilon}$$

$$\Phi_\varepsilon(t) - \int_0^t \frac{\sin(t-x)+1-\varepsilon \cos(t-x)}{\varepsilon} \Phi_\varepsilon(x)dx$$

$$= \frac{(\varepsilon-t-2)\sin(t-\varepsilon)}{2\varepsilon}$$

$$\Phi_\varepsilon(t) + \int_0^t \cos(t-x)\Phi_\varepsilon(x)dx$$

$$= \frac{(t+\varepsilon+2)\sin(t+\varepsilon) - (t-\varepsilon+2)\sin(t-\varepsilon)}{4\varepsilon}$$

for $t \in [0,1]$ with $\Phi_\varepsilon(t) = \cos(\phi(t))$.
The exact solution is:

$$\phi(t) = t.$$

Table 2. Comparison of the absolute errors for Example 2 of VK2_{fda}, VK2_{bda} and VK2_{cda} obtained by Taylor approximation ($\varepsilon = 10^{-4}$) and Nyström method ($n = 20$)

t_{2j} Simpson	Error of VK2 _{fda}	Error of VK2 _{cda}	Error of VK2 _{bda}
0	9.9998e-03	5.7735e-05	1.0000e-02
0.1	5.0009e-04	4.9477e-05	6.0614e-04
0.2	2.4721e-04	4.6874e-05	3.6699e-04
0.3	1.6198e-04	4.4267e-05	2.9405e-04
0.4	1.1847e-04	4.1656e-05	2.6364e-04
0.5	9.1614e-05	3.9037e-05	2.5100e-04
0.6	7.3061e-05	3.6406e-05	2.4800e-04
0.7	5.9226e-05	3.3757e-05	2.5130e-04
0.8	4.8310e-05	3.1084e-05	2.5938e-04
0.9	3.9311e-05	2.8379e-05	2.7155e-04
1	3.1619e-05	2.5632e-05	2.8759e-04
t_{2j} Trapezoidal	Error of VK2 _{fda}	Error of VK2 _{cda}	Error of VK2 _{bda}
0	9.9998e-03	5.7735e-05	1.0000e-02
0.1	1.5380e-04	3.9742e-04	1.0217e-03
0.2	1.0231e-04	3.7812e-04	7.9744e-04
0.3	1.9265e-04	3.6031e-04	7.4007e-04
0.4	2.4253e-04	3.4369e-04	7.2760e-04
0.5	2.7712e-04	3.2815e-04	7.3592e-04
0.6	3.0495e-04	3.1361e-04	7.5750e-04
0.7	3.2983e-04	3.0000e-04	7.8963e-04
0.8	3.5383e-04	2.8727e-04	8.3157e-04
0.9	3.7829e-04	2.7537e-04	8.8358e-04
1	4.0420e-04	2.6426e-04	9.4658e-04

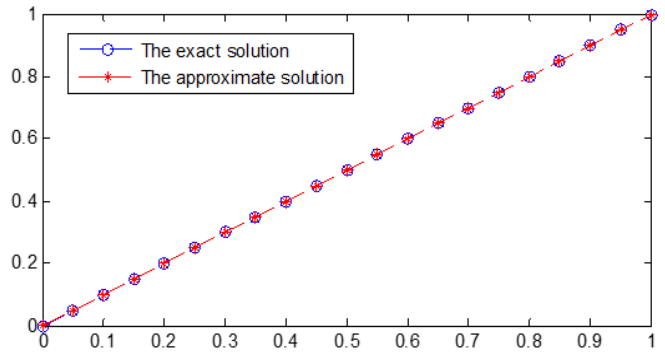
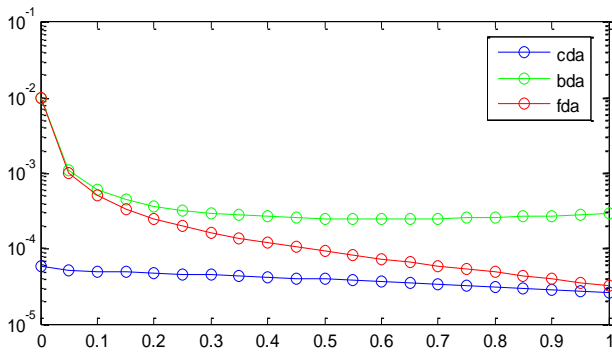


Figure 2. Comparison of resultants to Example 2, for $n=20$ and $\varepsilon = 10^{-4}$ by Simpson method

Example 3: [8]

Let be a Volterra -Hammerstein integral equation of the form:

$$\int_0^t e^{t-x} \phi^2(x) dx = e^{2t} - e^t, \quad 0 \leq x \leq t \leq 1$$

To solve this equation, first we convert it to linear VK2 of the second kind ($VK2_{fda}$, $VK2_{bda}$ and $VK2_{cda}$, respectively) of the flowing form:

$$\Phi_\varepsilon(t) + \int_0^t \frac{1+\varepsilon}{\varepsilon} e^{t-x} \Phi_\varepsilon(x) dx = \frac{e^{2(t+\varepsilon)} - e^{(t+\varepsilon)}}{\varepsilon}$$

$$\Phi_\varepsilon(t) + \int_0^t \frac{-1+\varepsilon}{\varepsilon} e^{t-x} \Phi_\varepsilon(x) dx = \frac{-e^{2(t-\varepsilon)} + e^{(-t+\varepsilon)}}{\varepsilon}$$

$$\Phi_\varepsilon(t) + \int_0^t e^{t-x} \Phi_\varepsilon(x) dx = \frac{e^{2(t+\varepsilon)} - e^{(t+\varepsilon)} - e^{2(t-\varepsilon)} + e^{(t-\varepsilon)}}{2\varepsilon}$$

for $t \in [0,1]$, $\Phi_\varepsilon(t) = \phi^2(t)$.
The exact solution is:

$$\phi(t) = e^t.$$

Table 3. Comparison of the absolute errors for Example 3 of $VK2_{fda}$, $VK2_{bda}$ and $VK2_{cda}$ obtained by Taylor approximation ($\varepsilon = 10^{-4}$) and Nyström method ($n = 20$)

t_{2j} Simpson	Error of $VK2_{fda}$	Error of $VK2_{cda}$	Error of $VK2_{bda}$
0	7.5003e-05	5.8333e-09	7.4997e-05
0.1	4.2112e-05	1.0427e-05	8.6028e-05
0.2	1.0755e-05	2.0965e-05	9.8212e-05
0.3	1.9516e-05	3.1712e-05	1.1167e-04
0.4	4.9126e-05	4.2777e-05	1.2653e-04
0.5	7.8474e-05	5.4270e-05	1.4295e-04
0.6	1.0795e-04	6.6307e-05	1.6109e-04
0.7	1.3792e-04	7.9007e-05	1.8112e-04
0.8	1.6877e-04	9.2497e-05	2.0326e-04
0.9	2.0086e-04	1.0691e-04	2.2771e-04
1	2.3456e-04	1.2240e-04	2.5473e-04
t_{2j} Trapezoid	Error of $VK2_{fda}$	Error of $VK2_{cda}$	Error of $VK2_{bda}$
0	7.5003e-05	5.8333e-09	7.4997e-05
0.1	6.1211e-05	1.0428e-05	8.5480e-05
0.2	4.6312e-05	2.0965e-05	9.7107e-05
0.3	3.0184e-05	3.1713e-05	1.1000e-04
0.4	1.2693e-05	4.2777e-05	1.2429e-04
0.5	6.3098e-06	5.4270e-05	1.4013e-04
0.6	2.6989e-05	6.6306e-05	1.5768e-04
0.7	4.9525e-05	7.9005e-05	1.7712e-04
0.8	7.4118e-05	9.2495e-05	1.9865e-04
0.9	1.0099e-04	1.0691e-04	2.2249e-04
1	1.3039e-04	1.2240e-04	2.4889e-04

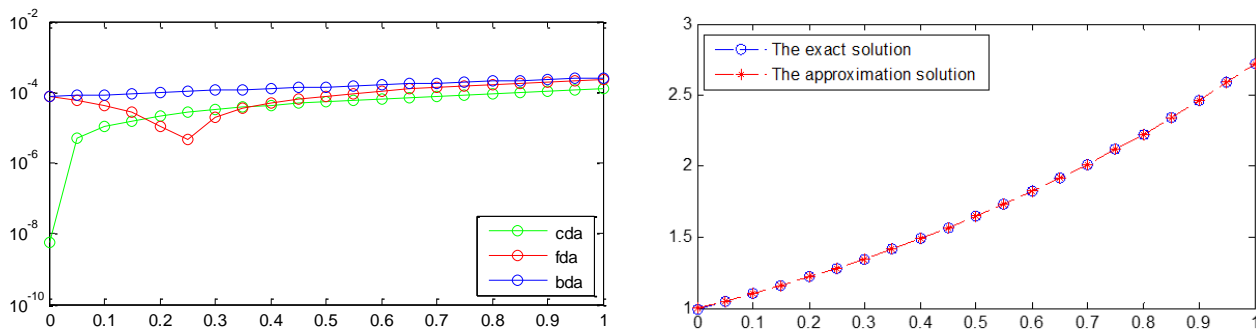


Figure 3. Comparison of resultants to Example 3, for $n=20$ and $\varepsilon = 10^{-4}$ by Simpson method

Table 4. Comparison of maximum absolute error of proposed algorithm “cda” for Example 3

n	Trapezoidal Method	Modified Simpson Method	Absolute Errors [8]
8	7.6466e-04	6.4495e-04	2.8 e-03
16	1.9124e-04	1.7592e-04	7.3 e-04
32	4.7807e-05	4.5869e-05	1.8 e-04
64	1.1945e-05	1.1701e-05	4.6 e-05
128	2.9787e-06	2.9482e-06	1.1 e-05
256	7.3719e-07	7.3340e-07	2.9 e-06
512	1.7681e-07	1.7635e-07	7.3 e-07

As expected, Tables 1-3 and Figures 1-3 demonstrate that the convergence rate of Nyström methods to "cda" approximation equation (if $\varepsilon \rightarrow 0$) is much faster and more accurate than "fda" and "bda". However, the convergence rate of the proposed algorithms is quicker.

Table 4 above displays the maximum absolute errors for various values of n . Furthermore, this final table presents a comparison of the absolute mistakes that are produced by the method that we propose with the one that is described by Inderdeep and Sheo [8] of Example 3.

The proposed approach has smaller absolute errors than the absolute errors presented by Inderdeep and Sheo [8]. This shows that proposed method “cda” is more accurate than the method presented by Inderdeep and Sheo [8] for ($\varepsilon=10^{-4}$).

4. CONCLUSIONS

In this paper, we gave an approach technique that uses Taylor series and Nyström methods for approximating solution of first kind nonlinear Volterra problems, the effectiveness of this above technique was tested by utilizing three distinct examples. It has been observed that all equivalent equations converge and the absolute error is near which was proved that numeric results were accepted for all types of the first kind Volterra-Hammerstein integral equation. Then, the most accurate approximation by Taylor series is the central difference approximation "cda".

In our future project, we will apply the techniques that have been proposed to general equations and systems of all ill-posed problems.

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NOMENCLATURE

VK1	Volterra integral equation of the first kind
VK2	Volterra integral equation of the second kind
bda	The backward difference approximation
fda	The forward difference approximation
cda	The central difference approximation
AS	The approximate solutions
Err	The absolute errors