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Application of Laplace Transform Method for Solving Weakly-Singular Integro-Differential Equations

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ABSTRACT

In this paper, the aim is to study the weakly singular integro-differential equations, typically its significance appears in various applications of engineering and science. The contributions of this paper have been presented through the study of existence, uniqueness, and different stability of solution of the weakly singular integro-differential equations, and solving this type of equations analytically. We propose an analytic method based on Laplace transform to solve the weakly singular integro-differential equations, whose advantages lie in simplicity application and obtaining exact solutions. Some suitable examples have been provided to better understand this work and the results of experiments exhibited the proposed method as a simple approach and superiority in accuracy and efficiency of solution.

1. INTRODUCTION

Weakly singular integro-differential equations (WSIDE) have been growing importance enormously as it plays a key role to represent many typically in phenomena of physics and engineering as mathematical modeling. For instance, diffusion of separated particles in turbulent fluids [1], elasticity and fracture in mechanics [2], biosciences [3], potential problems, thermal conductivity problems, materials, radiative equilibrium, the Dirichlet problems [4-6] and so on.

So a variety works focused its interest to tackle the equations of this form which are usually numerical methods, including an operational method [7], Bernstein series [8], Block boundary value method [9], Partition of the interval and introduction of additional parameters [10], Smoothing transformation and spline collocation [11], The asymptotic estimations of the solution [12], Product integration [13], collocations methods as Spline, Piecewise Polynomial, and Spectral respectively [14-16], but it is well known that the results of numerical methods have an error rate.

To address the WSIDE with different orders of derivatives without error rate in results, in this paper we propose analytic efficient and simple technique to yield exact solution which is Laplace transform method (LTM). Procedure of Laplace transform (LT) with differential equations and integral equations is changing them to polynomial equations and easily can be solved, and thus the solution of the considered equation is obtained by taking the inverse Laplace transform (LT) for it [17, 18]. Our motivation of this work is to present exact solution for many real-world problems are described as WSIDE by using efficient and simple method.

The remainder of this paper is organized as follows. In Section 2, the definition of concept WSIDE is expressed. In Section 3, we discuss the existence and uniqueness of solution for WSIDE. Section 4 discusses the various stability of solution for WSIDE. In Section 5, we introduce the concept of Laplace transform and it is followed up implementation it to solve WSIDE. In Section 6, the exact solutions of the WSIDE are obtained using the Laplace transform by solving several examples. Finally, conclusions are drawn in Section 7.

2. THE WEAKLY-SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS (WSIDE)

Consider the following standard form of mth-order linear integro-differential equation of the second kind:

$$y^{(m)}(t) = g(t) + \int_a^t k(s,t)\, y(s)ds, a \le t \le b, \quad 0 < m < \infty \tag{1}$$

with initial conditions $y^{(j)}(a) = y_j$, $j = 0, 1, \dots, m-1$, where $y^{(m)}(t) = \frac{d^m y}{dt^t}$, $m < \infty$, $g(t), k(s,t)$ are given functions and $y(t)$ is the unknown function, also $k(s,t)$ is denoted the kernel of the integro equation. We usually propose that the functions $y(t)$ and $g(t)$ are continuous or square integrable on $[a,b]$. Furthermore, Eq. (1) is also denoted a singular integral equation if the kernel $K(x, t)$ becomes infinite at one or more points in the domain of integration.

Motivation of present work is the desire to obtain exact analytic solution by using LTM for a linear weakly singular Volterra integro-differential equation (WSVIDE) of mth order and the second kind, where the integrand is denote as a weakly singular in the sense that its integral is continuous at the singular point, that is, its kernel $k(s,t) = \frac{1}{(t-s)^\alpha}$ is singular as

$s \to t$, where, $0 < \alpha < 1$ is positive real constant.

3. EXISTENCE AND UNIQUENESS SOLUTION OF WSIDE

This section discusses the existence and uniqueness of solution for the WSIDE and toward meeting that, it will be beneficial considering following procedure:

Let $y^{(m)}(t) = f(t)$, where $f(t)$ is a continuous real valued function defined on $[a, b]$. Integrating both sides m-times from a to t and using the initial conditions, yields to:

$$y(t) = \sum_{j=0}^{m-1} \frac{(t-a)^j}{j!} y_j \int_a^t \frac{(t-x)^{m-1}}{(m-1)!} f(x)dx \tag{2}$$

Eq. (1) can be written as:

$$\begin{aligned} f(t) &= g(t) + \int_a^t k(s,t) \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \left(\int_a^s \frac{(s-x)^{m-1}}{(m-1)!} f(x)dx \right) ds \\ &= g(t) + \int_a^t \frac{k(s,t)}{(m-1)!} \left(\sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right) \left(\int_a^s (s-x)^{m-1} f(x)dx \right) ds \end{aligned} \tag{3}$$

Suppose,

$$W(t,s) = \frac{k(s,t)}{(m-1)!} \left(\sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right)$$

So that Eq. (3) is transformed into,

$$f(t) = g(t) + \int_a^t W(t,s) \left(\int_a^s (s-x)^{m-1} f(x)dx \right) ds \tag{4}$$

Now, having the existence theory of solution at our disposal, we discuss it.

Theorem 1. Assume that $f \in C([a,b], R), k \in C([a,b] \times [a,b], R)$ for $a \le s < t \le b$ are continuous, where β_1 upper bound of $|k(t,s)|, \forall\, (t,s) \in [a,b]^2$, $\beta_2 = \frac{(t-a)^m}{m!}$ and $\beta_3 = \left| \sum_{j=0}^{m-1} \frac{(t-a)^j}{j!} y_j \right|$. Then the WSIDE in Eq. (1) satisfies the following:

i. It has at least one continuous solution.
ii. It has a unique continuous solution.

Proof 1. Let T be an integral operator defined by:

$$Tf(t) = g(t) + \int_a^t W(t,s) \left(\int_a^s (s-x)^{m-1} f(x)dx \right) ds \tag{5}$$

from Eqs. (4) and (5) we have:

$$f = T(f)$$

consider the set $\Omega_r = \{f \in C([a,b], R) : \|f\|_\infty = sup_{t\in[a,b]}|f| \le r\}$. The radius r is the solution of an equation $\beta_1(t-a)(\beta_2 r + \beta_3) + \|f\|_\infty = r$, we have to prove the operator T maps the set Ω_r into itself, to observe that, let f be any function in Ω_r. Then:

$$\begin{aligned} |T(f)(t)| &\le |g(t)| + \int_a^t |W(t,s)| \left(\int_a^s (s-x)^{m-1} f(x)dx \right) ds \\ &\le \|g(t)\|_\infty + \int_a^t \frac{r(t-a)^m |k(t,s)|}{m!} \left| \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right| ds \end{aligned}$$

since $k(t,s)$ is continuous on $[a,b]^2$, therefore, $k(t,s)$ is bounded and hence $|\mathrm{k(t,s)}| \le \beta_1, \forall\, (t,s) \in [a,b]^2, \beta_1 > 0$.

$$|T(f)(t)| \le \|g(t)\|_\infty + (t-a)\beta_1 \left(\frac{r(t-a)^m}{m!} \right) \left| \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right|$$

This implies that:

$$\begin{aligned} |T(f)(t)| &\le \|g(t)\|_\infty + \beta_1(t-a)(r\beta_2 + \beta_3) \\ &= \|g(t)\|_\infty + r - \|g(t)\|_\infty = r \end{aligned} \tag{6}$$

Thus, $\|T(f)\|_\infty \le r$. Suppose $t_1, t_2 \in [a,b]$, and without loss of generality, let $t_1 < t_2$. Assume $f \in \Omega_r$.

$$\begin{aligned} &|T(f)(t_2) - T(f)(t_1)| \le |g(t_2) - g(t_1)| \\ &+ \int_a^t |W(t_2,s) - W(t_1,s)| \left(\int_a^s (s-x)^{m-1} f(x)dx \right) ds \end{aligned} \tag{7}$$

But the functions g and W are continuous. So, we have:

$$\begin{gathered} |g(t_2) - g(t_1)| \to 0 \\ |W(t_2,s) - W(t_1,s)| \to 0 \end{gathered}$$

When $t_2 \to t_1$ and thus $|T(f)(t_2) - T(f)(t_1)| \to 0$. So, the operator T maps the set Ω_r into itself. Next, to prove the existence of fixed point of T in Ω_r, we may apply Schauder's fixed point theorem which is equivalent to solving Eq. (1), as follows. Let $\{f_i\}_{i=1}^\infty$ be a sequence with $f_i \in \Omega_r$ and suppose $f_i \to f \in \Omega_r$, when $i \to \infty$. Clearly that $\|f_i\|_\infty \le r$, for all $i \in N$.

$$\begin{aligned} &|T(f_i)(t) - T(f)(t)| \le \\ &\int_a^t |W(t,s)| \left| \left(\int_a^s (s-x)^{m-1} f_i(x)dx - \int_a^s (s-x)^{m-1} f(x)dx \right) \right| ds \\ &\le \int_a^t |W(t,s)| \left(\int_a^s (s-x)^{m-1} |f_i(x) - f(x)| dx \right) ds \end{aligned}$$

Hence by Arzela bounded convergence theorem gives:

$$\begin{aligned} &\lim_{i\to\infty} \int_a^t |W(t,s)| \left(\int_a^s (s-x)^{m-1} |f_i(x) - f(x)| dx \right) ds \\ &= \int_a^t |W(t,s)| \left(\int_a^s (s-x)^{m-1} \lim_{i\to\infty} |f_i(x) - f(x)| dx \right) ds = 0 \end{aligned}$$

Thus, $|T(f_i)(t) - T(f)(t)| \to 0$, when $i \to 0$ and we can conclude the operator T is continuous. Letting $(T(f_i))$ be a sequence for any $f_i \in \Omega_r$. From Eq. (6), $\|T(f_i)\|_\infty \le r$, for all $i \in N$, we deduce the sequence $(T(f_i))$ is uniformly bounded on $[a,b]$. Moreover, from inequality (7) that for any ϵ there exist δ such that $|T(f_i)(t_2) - T(f_i)(t_1)| < \epsilon$ provided that $|t_2 - t_1| < \delta$ for all $i \in N$, we conclude the sequence $(T(f_i))$

is an equicontinuous. By the Ascoli-Arzela theorem [19], there exist subsequence $\left(T(f_{i_k})\right) \subset (T(f_i)), k \to \infty$ is convergence uniformly. Consequently, the set $T(\Omega_r)$ is compact, which implies T is completely continuous operator. Hence by Schauder fixed point theorem [17], Eq. (4) has a fixed point in Ω_r and implies that an integro-differential equation Eq. (1) possesses at least one solution.

Proof 2. Let f and f^* be any two solutions of Eq. (4), then

$$\begin{aligned} &|f(t) - f^*(t)| \leq \\ &\int_a^t |W(t,s)| \left| \left(\int_a^s (s-x)^{m-1} f(x)dx \right.\right. \\ &\qquad \left.\left. - \int_a^s (s-x)^{m-1} f^*(x)dx \right) \right| ds \\ &\leq \beta_1\beta_2(t-a)\beta_3 \left\| f(t) - f^*(t) \right\|_\infty \\ &\leq (t-a)\beta_1\beta_2\beta_3 \left\| f(t) - f^*(t) \right\|_\infty \end{aligned} \tag{8}$$

by passing supremum with respect to $t \in [a,b]$ for both side of inequality (8) gives:

$$(1 - (t-a)\beta_1\beta_2\beta_3 \|f(t) - f^*(t)\|_\infty) < 0$$

but we have $(t-a)\beta_1\beta_2\beta_3 < 1$. Therefore, we must have $\|f(t) - f^*(t)\|_\infty = 0$ and this proves that $f(t) = f^*(t)$.

Hence the Eq. (1) has a unique solution on $[a,b]$.

4. STABILITY SOLUTION OF WSIDE

In this section we investigate the stability of the solution for the WSIDE, for this using the idea of the variation of parameters formula for linear differential systems to obtain an integral equation for the solutions of Eq. (1). For this purpose, let $X(t)$ be a fundamental matrix solution of:

$$y^{(m)} = Ay,$$

so that any solution of Eq. (1) with the initial functions ψ on $[t_0, r]$ is given by:

$$\begin{aligned} y(t,\tau,\psi) &= X(t)X^{-1}(\tau) \\ &+ \frac{1}{(m-1)!} \int_\tau^t X(t)X^{-1}(s)(t \\ &- s)^{m-1}[\int_a^t k(s,t)\, y(s)ds + f(t)]ds \end{aligned} \tag{9}$$

and prove stability results for the system (1). We begin with giving a suitable definition of stability, then lemma and the theorems.

Definition 1 [19]: The solution of Eq. (1) is said to be stable if there exists $\epsilon > 0$ and $t_0 \in R_+$, given a $\delta = \delta(t_0, \epsilon) > 0$ such that $|\psi|_{t_0} = \max_{0<s\leq t} |\psi(s)| < \delta$ implies $|y(t)| < \epsilon, t \geq 0$.

Lemma 1 [20]: (Gronwall's inequality) Let $y(t), x(t) \in C([a,+\infty), R_+)$, $K > 0$ is constant and suppose that:

$$y(t) \leq K + \int_a^t y(s)x(s)ds, \quad s \in [a,+\infty)$$

then

$$y(t) \leq K\, exp(\int_a^t x(s)ds), \quad s \in [a,+\infty)$$

Theorems 2: Assume that:

$$|X(t)X^{-1}(s)| \leq K, \quad 0 \leq s \leq t,$$

$$\int_0^\infty \int_0^t \left| \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \right| dsdt \leq M_1$$

and

$$\int_0^\infty \left| \frac{(t-s)^{m-1}}{(m-1)!} f(t) \right| dt \leq M_2$$

Then the solution of Eq. (1) is uniformly stable.

Proof: Set $\delta(\epsilon) < \frac{\epsilon}{Ke^{KM}}$ and $\|\psi\|_{t_0} < \delta(\epsilon)$ for any $\epsilon < 0$. Assume that there exists $t_0 \leq t_1$ such that $|y(t_1)| = \epsilon$ and $|y(t)| = \epsilon$ on $[t_0, t_1)$. By using Eq. (9), we get:

$$\begin{aligned} |y(t)| &\leq |X(t)X^{-1}(t_0)||\psi(t_0)| \\ &+ \frac{1}{(m-1)!} \int_{t_0}^t |X(t)X^{-1}(s)|\,|(t \\ &- s)^{m-1}| \left[\int_a^s |k(s,t)|\,|y(s)|du + f(t) \right] ds \end{aligned}$$

then

$$\begin{aligned} &|y(t)| \leq K\,\delta(\epsilon) + \\ &K \int_{t_0}^t \frac{|(t-s)^{m-1}|}{(m-1)!} \left[\int_a^s |k(s,t)|\,|y(s)|du + |f(t)| \right] ds \end{aligned}$$

on $[t_0, t_1]$ and define $r(t) \equiv \max_{0<s\leq t} |y(s)|$ to obtain:

$$\begin{aligned} &r(t) \leq K\,\delta(\epsilon) + \\ &K \int_{t_0}^t \frac{|(t-s)^{m-1}|}{(m-1)!} \left[(\int_a^s |k(s,t)|\,du) r(s) + |f(t)| \right] ds \end{aligned}$$

by Lemma 1 (Gronwall's inequality), then we get:

$$\begin{aligned} &|y(t)| \leq r(t) \leq K\,\delta(\epsilon) \\ &exp\,[K \int_{t_0}^t \frac{|(t-s)^{m-1}|}{(m-1)!} \left(\int_a^s |k(s,t)|\,du + |f(t)| \right) ds] \end{aligned}$$

$$|y(t)| \leq r(t) \leq K\,\delta(\epsilon) exp\,[\mathrm{K}(\mathrm{M}_1 + \mathrm{M}_2)]$$

$$|y(t)| \leq r(t) \leq K\, e^{KM} \delta(\epsilon) < \epsilon \qquad \text{on } [t_0, t_1]$$

where, $\mathrm{M} = \mathrm{M}_1 + \mathrm{M}_2$.

Consequently $|y(t_1)| < \epsilon$ which is a contradiction. Thus, the solution of Eq. (1) is uniformly stable, completing the proof.

Theorems 3: Assume that:

$$\int_0^t |X(t)X^{-1}(s)|ds \le L\,, \quad \text{for } t \ge 0, \tag{10}$$

$$\sup_{t\ge 0} \int_0^t \left|\frac{(t-s)^{m-1}}{(m-1)!}K(s,t)\right| dt < \frac{1}{L} \tag{11}$$

and

$$\sup_{t\ge 0} \int_0^\infty \left|\frac{(t-s)^{m-1}}{(m-1)!}f(t)\right| dt \le M$$

Moreover, assume also that:

$$\lim_{s\to\infty} \int_0^t \left|\frac{(t-s)^{m-1}}{(m-1)!}K(t,u)\right| du = 0 \qquad \text{for all } t \ge 0$$

Then the solution of Eq. (1) is asymptotically stable.

Proof: We first show that stability of the solution. From Eq. (11), there exists a positive constant γ such that:

$$\sup_{t\ge 0} \int_0^t \left|\frac{(t-s)^{m-1}}{(m-1)!}K(s,t)\right| dt < \gamma$$

where, $0 < \gamma < \frac{1}{L}$

From inequality (10), there exists a positive constant N such that:

$$|X(t)| \le N \quad \text{for all } t \ge 0$$

For any $\epsilon > 0$ and $t \ge 0$, let $\delta = \delta(\epsilon, t_0) < \min\{\frac{\left(\frac{1}{2}-\gamma L\right)\epsilon}{(N|X^{-1}(t_0)|)}, \epsilon\}$.

Consider the solution of Eq. (1) such that $|\psi|_{t_0} < \delta$. Suppose there exists $t_1 > t_0$ such that $|y(t_1)| = \epsilon$, $|y(t)| < \epsilon$ on $[t_0, t_1)$ and $M \le \frac{\epsilon}{2L}$. For all $t \in [t_0, t_1]$, we have:

$$\begin{aligned}|y(t)| &\le |X(t)X^{-1}(t_0)||\psi(t_0)| \\ &+ \frac{1}{(m-1)!}\int_{t_0}^t |X(t)X^{-1}(s)|\,|(t-s)^{m-1}| \\ &\left[\int_a^s |k(s,t)|\,|y(s)|du + |f(t)|\right] ds\end{aligned}$$

leads to

$$\begin{aligned}|y(t)| &< N|X^{-1}(t_0)|\,\delta + L\int_{t_0}^t \frac{|(t-s)^{m-1}|}{(m-1)!} \\ &\left[\int_a^s |k(s,t)|\,|y(s)|du + |f(t)|\right] ds \\ &< \left(\frac{1}{2} - \gamma L\right)\epsilon + L\left(\gamma\epsilon + \frac{\epsilon}{2L}\right) = \epsilon\end{aligned}$$

Thus, $|y(t_1)| < \epsilon$, which is a contradiction. Thus, the solution of Eq. (1) is asymptotically stable.

Theorems 4: Suppose that there exist positive constants $K \ge 1$, λ and μ such that:

$$|X(t)X^{-1}(s)| \le Ke^{-\lambda(s-t)} \quad \text{for } 0 \le s \le t < \infty \tag{12}$$

$$\sup_{t\ge 0} \int_0^t e^{-\mu(s-t)}\left|\frac{(t-s)^{m-1}}{(m-1)!}K(s,t)\right| dt < \frac{\lambda}{K} \tag{13}$$

and

$$\sup_{t\ge 0} \int_0^\infty e^{-\mu(s-t)}\left|\frac{(t-s)^{m-1}}{(m-1)!}f(t)\right| dt \le M$$

Then the solution of Eq. (1) is exponentially asymptotically stable.

Proof: For all $t \ge t_0$ and $|\psi|_{t_0} < \frac{1}{K}$, we have

$$\begin{aligned}|y(t)| &\le Ke^{-\lambda(t-t_0)}|\psi(t_0)| \\ &+ K\int_{t_0}^t e^{-\lambda(s-t)}\frac{|(t-s)^{m-1}|}{(m-1)!} \\ &\left[\int_a^s |k(s,t)|\,|y(s)|du + f(t)\right] ds\end{aligned} \tag{14}$$

There exist positive constants $v < \mu$ and σ such that $\lambda = v + \sigma$ and $\sup_{t\ge 0} \int_0^t e^{v(s-t)}\left|\frac{(t-s)^{m-1}}{(m-1)!}K(s,t)\right| dt < \frac{\sigma}{K}$. Multiply both sides of inequality (14) by e^{vt} to obtain:

$$\begin{aligned}e^{vt}|y(t)| &\le Ke^{vt}e^{-\sigma(t-t_0)}|\psi(t_0)| + K\int_{t_0}^t e^{vt}e^{-\sigma(s-t)} \\ &\frac{|(t-s)^{m-1}|}{(m-1)!}\left[\int_a^s |k(s,u)|\,|y(u)|du + f(t)\right] ds \\ &= Ke^{vt_0}e^{-\sigma(t-t_0)}|\psi(t_0)| + K\int_{t_0}^t e^{-\sigma(s-t)}\frac{|(t-s)^{m-1}|}{(m-1)!} \\ &\left[\int_a^s e^{v(s-u)}|k(s,u)|\,e^{vu}|y(u)|du + e^{vt}f(t)\right] ds\end{aligned}$$

Let denote $\sup_{t\ge 0} e^{vs}|y(s)|$ by $r(t)$, it follows that:

$$\begin{aligned}e^{vt}|y(t)| &\le Ke^{vt_0}e^{-\sigma(t-t_0)}|\psi(t_0)| + (\sigma r(t) \\ &+ M)\int_{t_0}^t e^{-\sigma(t-s)}\,ds \\ &\le Ke^{vt_0}e^{-\sigma(t-t_0)}|\psi(t_0)| + \\ &\sigma r(t)\int_{t_0}^t e^{-\sigma(t-s)}\,ds + M\int_{t_0}^t e^{-\sigma(t-s)}\,ds \\ &\le Ke^{vt_0}e^{-\sigma(t-t_0)}|\psi(t_0)| + \\ &(1-e^{-\sigma(t-t_0)})r(t) + Me^{-\sigma(t-t_0)}\end{aligned}$$

From above inequality, if $e^{vs}|y(s)| < e^{vt}|y(t)|$ for any $s \in [0,t]$, we see that $r(t) = e^{vt}|y(t)|$, we get:

$$\begin{aligned}r(t) &\le Ke^{vt_0}e^{-\sigma(t-t_0)}|\psi(t_0)| + (1-e^{-\sigma(t-t_0)})r(t) \\ &+ Me^{-\sigma(t-t_0)}\end{aligned}$$

Therefore, $r(t) \le Ke^{vt_0}|\psi(t_0)|$ for $t \ge t_0$. Then $r(t) = e^{vt}|y(t)|$ implies $|y(t)| \le Ke^{-v(t-t_0)}|\psi(t_0)|$ for $t \ge t_0$.

If there exists $s \in [0,t]$ such that $e^{vs}|y(s)| > e^{vt}|y(t)|$, we have the following two further cases:

(a) There exists $t_1 \in [t_0, t]$ such that $e^{vt_1}|y(t_1)| = r(t)$. Then from equation (****) we have:

$$r(t_1) = e^{vt_1}|y(t_1)| \le Ke^{vt_0}e^{-\sigma(t_1-t_0)}|\psi(t_0)| + (1 - e^{-\sigma(t_1-t_0)})r(t_1) + Me^{-\sigma(t_1-t_0)}$$

Thus $r(t_1) \le Ke^{vt_0}|\psi(t_0)|$ for $t_1 \ge t_0$. Then $e^{vt}|y(t)| < r(t_1)$ implies $|y(t)| \le Ke^{-v(t-t_0)}|\psi(t_0)|$ for $t \ge t_0$.

(b) There exists $t_2 \in [0, t_0)$ such that $e^{vt_2}|y(t_2)| = r(t)$. Then $e^{vt}|y(t)| < e^{vt_2}|y(t_2)| < e^{vt_0}|\psi|_{t_0}$, we have $|y(t)| \le |\psi|_{t_0}e^{-v(t-t_0)}$.

Thus from (a) and (b), the solution of Eq. (1) is exponentially stable.

5. LAPLACE TRANSFORM METHOD (LTM) AND IMPLEMENTATION

5.1 Laplace transform method (LTM)

In general, the idea of a transformation is a very important in problem solving, where the difficult problem is changed in some way into an easier problem and then solve that easier problem to obtain solution and apply it to original problem.

The Laplace transform method (LTM) aims to seek solve the differential equations and integral equations easily by changing it to polynomial, and then taking the inverse Laplace transform, which lead for the solution of intended equation. Here we present a basic concept of LTM, for a given function $y(s)$ with respect $s \ge 0$, the Laplace transform method can be described as:

$$Y(\rho) = L\{y(s)\} = \int_0^\infty e^{-\rho s}\, y(s)ds$$

where, ρ is real, and L denotes the Laplace transform operator. Furthermore, a vanishment $Y(\rho)$ as ρ approaches infinity is an important and necessary condition for the existence of the Laplace transform $Y(\rho)$. This means that:

$$\lim_{\rho\to\infty} Y(\rho) = 0$$

As well as, there are key properties of the Laplace transforms which are used in the proposed framework are given briefly as follows:

1. Constant multiple:

$$L\{\alpha\, y(s)\} = \alpha\, L\{y(s)\},\ \ \alpha \text{ is constant.}$$

2. Linearity property: α, β are constant.

$$L\{\alpha\, y(s)\} + L\{\beta\, g(s)\} = \alpha\, L\{y(s)\}\,\beta\, L\{\, g(s)\},$$

3. Multiplication by s:

$$L\{s\, y(s)\} = \frac{d}{d\rho}\, L\{y(s)\} = -Y'(\rho)$$

4. Laplace transforms of derivatives:

$$L\{\, y'(s)\} = \rho L\{y(s)\} - y(0),$$

$$L\{\, y''(s)\} = \rho^2 L\{y(s)\} - \rho y(0) - y'(0),$$

$$L\{\, y'''(s)\} = \rho^3 L\{y(s)\} - \rho^2 y(0) - \rho y'(0) - y''(0),$$

$$\vdots$$

$$L\{\, y^{(n)}(s)\} = \rho^n L\{y(s)\} - \rho^{n-1}y(0) \cdots - \rho y^{n-2}(0) - y^{n-1}(0),$$

5. Inverse Laplace transform:

$$L^{-1}\{Y(\rho)\} = y(s)$$

6. Convolution theorem for Laplace transform:

$$L\{(y_1 * y_2)(s)\} = L\left\{\int_0^s y_1(s-t)y_2(t)dt\right\} = Y_1(\rho)Y_2(\rho)$$

There are also elementary Laplace transforms, some of which can be summarized briefly in Table 1.

Table 1. Some of elementary Laplace transforms

$\boldsymbol{y(s)}$	$\boldsymbol{Y(\rho) = L\{y(s)\} = \int_0^\infty e^{-\rho s}\, y(s)ds}$
k	$\frac{k}{\rho},\quad \rho > 0$
s	$\frac{1}{\rho^2},\quad \rho > 0$
s^n	$\frac{n!}{\rho^{n+1}} = \frac{\Gamma(n+1)}{\rho^{n+1}},\quad \rho > 0,\ \ Re\, n > -1$
e^{as}	$\frac{1}{\rho - a},\quad \rho > a$
$\delta(s-a)$	$e^{-a\rho},\quad a \ge 0$

5.2 Implementation

We now describe the implementation of the LTM to solve the consider WSVIDE, we can rewrite the expression for Eq. (1). After $k(t,s)$ was consistently set to be $k(t,s) = \frac{1}{(t-s)^\alpha}$, as follows:

$$y^{(m)}(s) = f(s) + \int_0^s \frac{1}{(t-s)^\alpha}\, y(t)dt \tag{15}$$

First, taking LT for both sides of Eq. (15), leading to:

$$L\{y^{(m)}(s)\} = L\{f(s)\} + L\{\int_0^s \frac{1}{(t-s)^\alpha}\, y(t)dt\} \tag{16}$$

by applying the convolution property of the LTM, Eq. (16) becomes:

$$L\{y^{(m)}(s)\} = L\{f(s)\} + L\{s^{-\alpha}\}L\{y(s)\}$$

so that:

$$\rho^m Y(\rho) - \rho^{m-1} y(0) - \cdots - \rho y^{(m-2)}(0) - y^{(m-1)}(0) = F(\rho) + \frac{\Gamma(1-\alpha)}{\rho^{1-\alpha}} Y(\rho)$$

$$Y(\rho) = \frac{\rho^{1-\alpha}\{F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0{}^{(m-2)} + y_0{}^{(m-1)}\}}{\rho^{m-1-\alpha} - \Gamma(1-\alpha)}$$

we take the inverse LT for both sides,

$$L^{-1}\{Y(\rho)\} = L^{-1}\{\frac{\rho^{1-\alpha}\{F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0{}^{(m-2)} + y_0{}^{(m-1)}\}}{\rho^{m-1-\alpha} - \Gamma(1-\alpha)}\}$$

hence, we find the solution of the Eq. (2) as:

$$y(s) = L^{-1}\{\frac{\rho^{1-\alpha}\{F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0{}^{(m-2)} + y_0{}^{(m-1)}\}}{\rho^{m-1-\alpha} - \Gamma(1-\alpha)}\}$$

6. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following linear first-order WSVID

$$y'(s) = -2\sqrt{s} + \int_0^s \frac{1}{\sqrt{s-t}} y(t)dt \tag{17}$$

with initial condition $y(0) = 0$, and exact solution $y(s) = 1$. By taking LT on both sides of Eq. (17), we obtain:

$$L\{y'(s)\} = L\{-2\sqrt{s}\} + L\{\int_0^s \frac{1}{\sqrt{s-t}} y(t)dt\}$$

by convolution theorem, give:

$$L\{y'(s)\} = L\{-2\sqrt{s}\} + L\{s^{-\frac{1}{2}}\} L\{y(s)\}$$

using elementary Laplace transform

$$\rho Y(\rho) - y(0) = \frac{\Gamma\left(\frac{3}{2}\right)}{\rho^{\frac{3}{2}}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{\frac{1}{2}}} Y(\rho)$$

substituting the given initial condition and simplistic:

$$Y(\rho)(\frac{\rho^{\frac{3}{2}} - \sqrt{\pi}}{\rho^{\frac{1}{2}}}) = \frac{\rho^{\frac{3}{2}} - \sqrt{\pi}}{\rho^{\frac{3}{2}}}$$

solving for $Y(\rho)$:

$$Y(\rho) = \frac{1}{\rho}$$

we take the inverse LT for both sides:

$$L^{-1}\{Y(\rho)\} = L^{-1}\{\frac{1}{\rho}\}$$

thus, the solution is $y(s) = 1$.

Example 2. Consider the following linear second-order WSVID:

$$y''(s) = -\frac{36}{55} s^{\frac{11}{6}} + \int_0^s \frac{1}{(s-t)^{\frac{1}{6}}} y(t)dt \tag{18}$$

with initial conditions $y'(0) = 0$, $y(0) = 0$, and the exact solution $y(s) = s$. Applying the LT on both sides of Eq. (18), we get:

$$L\{y''(s)\} = L\{-\frac{36}{55} s^{\frac{11}{6}}\} + L\{\int_0^s \frac{1}{(s-t)^{\frac{1}{6}}} y(t)dt\}$$

by convolution theorem, give:

$$L\{y''(s)\} = L\{-\frac{36}{55} s^{\frac{11}{6}}\} + L\{s^{\frac{-1}{6}}\} L\{y(t)\}$$

using elementary Laplace transform, this implies:

$$\rho^2 Y(\rho) - \rho y(0) - y'(0) = -\frac{\Gamma\left(\frac{5}{6}\right)}{\rho^{\frac{17}{6}}} + \frac{\Gamma\left(\frac{5}{6}\right)}{\rho^{\frac{5}{6}}} Y(\rho)$$

substitute the given initial conditions and simplistic:

$$Y(\rho)(\frac{\rho^{\frac{17}{6}} - \Gamma\left(\frac{5}{6}\right)}{\rho^{\frac{5}{6}}}) = \frac{\rho^{\frac{17}{6}} - \Gamma\left(\frac{5}{6}\right)}{\rho^{\frac{17}{6}}}$$

solving for $Y(\rho)$:

$$Y(\rho) = \frac{1}{\rho^2}$$

we take the inverse LT for both sides:

$$L^{-1}\{Y(\rho)\} = L^{-1}\{\frac{1}{\rho^2}\}$$

thus, the solution is $y(s) = s$.

Example 3. Consider the following linear third-order WSVID:

$$y'''(s) = -\frac{4}{3} s^{\frac{3}{2}} + \int_0^s \frac{1}{\sqrt{s-t}} y(t)dt \tag{19}$$

with initial conditions $y''(0) = 0$, $y'(0) = 1$, $y(0) = 0$, and the exact solution $y(s) = s$. By taking LT on both sides of Eq. (19), we obtain:

$$L\{y'''(s)\} = L\{-\frac{4}{3} s^{\frac{3}{2}}\} + L\{\int_0^s \frac{1}{\sqrt{s-t}} y(t)dt\}$$

Using convolution theorem, gives:

$$L\{y'''(s)\} = L\{-\frac{4}{3} s^{\frac{3}{2}}\} + L\{s^{\frac{-1}{2}}\} L\{y(t)\}$$

which becomes:

$$\rho^3 Y(\rho) - \rho^2 y(0) - \rho y'(0) - y''(0) = -\frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{\frac{5}{2}}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{\frac{1}{2}}} Y(\rho)$$

substitute the given initial conditions and simplistic:

$$Y(\rho)(\frac{\rho^{\frac{7}{2}}-\sqrt{\pi}}{\rho^{\frac{1}{2}}})=\frac{\rho^{\frac{7}{2}}-\sqrt{\pi}}{\rho^{\frac{5}{2}}}$$

solving for $Y(\rho)$:

$$Y(\rho)=\frac{1}{\rho^2}$$

we take the inverse LT for both sides:

$$L^{-1}\{Y(\rho)\}=L^{-1}\{\frac{1}{\rho^2}\}$$

thus, the solution is $y(s)=s$.

Example 4. Consider the following linear fourth-order WSVIDE

$$y''''(s)=-\frac{128}{45}s^{\frac{9}{4}}+\int_0^s\frac{1}{(s-t)^{\frac{3}{4}}}y(t)dt \tag{20}$$

with initial conditions $y'''(0)=0$ $y''(0)=2$, $y'(0)=0$, $y(0)=0$, and the exact solution $y(s)=s^2$. By taking LT on both sides of Eq. (20), we get:

$$L\{y''''(s)\}=L\{-\frac{128}{45}s^{\frac{9}{4}}\}+L\{\int_0^s\frac{1}{(s-t)^{\frac{3}{4}}}y(t)dt\}$$

then using convolution theorem leads us to conclude that:

$$L\{y''''(s)\}=L\{-\frac{128}{45}s^{\frac{9}{4}}\}+L\{s^{\frac{-3}{4}}\}L\{y(t)\}$$

this simplifies to:

$$\rho^4Y(\rho)-\rho^3y(0)-\rho^2y'(0)-\rho y''(0)-y'''(0) = -\frac{2\Gamma\left(\frac{1}{4}\right)}{\rho^{\frac{13}{4}}}+\frac{\Gamma\left(\frac{1}{4}\right)}{\rho^{\frac{1}{4}}}Y(\rho)$$

substitute the given initial conditions and simplistic:

$$Y(\rho)(\frac{\rho^{\frac{17}{4}}-\Gamma\left(\frac{1}{4}\right)}{\rho^{\frac{1}{4}}})=\frac{2(\rho^{\frac{17}{4}}-\Gamma\left(\frac{1}{4}\right))}{\rho^{\frac{13}{4}}}$$

solving for $Y(\rho)$:

$$Y(\rho)=\frac{2}{\rho^3}$$

we take the inverse LT for both sides:

$$L^{-1}\{Y(\rho)\}=L^{-1}\{\frac{2}{\rho^3}\}$$

thus, the solution is $y(s)=s^3$.

7. CONCLUSIONS

In this work, we have proposed a study for the weakly singular integro-differential equations by introducing a theory framework that has been achieved by proving the theorems of an existence, uniqueness, and stabilities of solution.

Along with that, we have presented a simple and powerful tool to solve WSIDE which is the Laplace transform technique that was investigated to solve WSIDE for different orders. We have succeeded to obtain the exact solutions by testing our proposed method on different examples of different orders. Encouragingly for next work, the proposed method will be used to solve the system of WSIDE.

REFERENCES

[1] Brunner, H., Tang, T. (1989). Polynomial spline collocation methods for the nonlinear Basset equation. Computers & Mathematics with Applications, 8(5): 449-457. https://doi.org/10.1016/0898-1221(89)90239-3

[2] Zozulya, V.V., Gonzalez-Chi, P.I. (1999). Weakly singular, singular and hypersingular integrals in 3-D elasticity and fracture mechanics. Journal of the Chinese Institute of Engineers, 22(6): 763-775. https://doi.org/10.1080/02533839.1999.9670512

[3] Brunner, H. (2004). Collocation Methods for Volterra Integral and Related Functional Differential Equations. Cambridge University Press.

[4] Brewer, D.W., Powers, R.K. (1990). Parameter identification in a Volterra equation with weakly singular kernel. The Journal of Integral Equations and Applications, 1: 353-373. http://doi.org/10.1216/jiea/1181075568

[5] Tang, B.Q., Li, X.F. (2008). Solution of a class of Volterra integral equations with singular and weakly singular kernels. Applied Mathematics and Computation, 199(2): 406-413. https://doi.org/10.1016/j.amc.2007.09.058

[6] Kythe, P., Puri, P. (2002). Computational Methods for Linear Integral Equations. Springer Science & Business Media. http://doi.org/10.1007/978-1-4612-0101-4

[7] Sadri, K., Ayati, Z. (2018). A new operational approach for solving weakly singular integro-differential equations. Hacettepe Journal of Mathematics and Statistics, 47(1): 107-127. http://doi.org/10.15672/HJMS.2017.519

[8] Işik, O.R., Sezer, M., Güney, Z. (2011). Bernstein series solution of a class of linear integro-differential equations with weakly singular kernel. Applied Mathematics and Computation, 217(16): 7009-7020. https://doi.org/10.1016/j.amc.2011.01.114

[9] Zhou, Y., Stynes, M. (2022). Block boundary value methods for solving linear neutral Volterra integro-differential equations with weakly singular kernels. Journal of Computational and Applied Mathematics, 401(1): 1137-1147. https://doi.org/10.1016/j.cam.2021.11347

[10] Assanova, A.T., Nurmukanbet, S.N. (2022). A solvability of a problem for a Fredholm integro-differential equation with weakly singular kernel. Lobachevskii Journal of Mathematics, 43: 182-191. https://doi.org/10.1134/S1995080222040047

[11] Diogo, T., Lima, P.M., Pedas, A., Vainikko, G. (2017). Smoothing transformation and spline collocation for

weakly singular Volterra integro-differential equations. Applied Numerical Mathematics, 114: 63-76. https://doi.org/10.1016/j.apnum.2016.08.009
[12] Dauylbaev, M.K., Mirzakulova, A.E. (2016). Asymptotic behavior of solutions of singular integro-differential equations. Discontinuity, Nonlinearity, and Complexity, 5(2): 145-152. http://doi.org/10.5890/DNC.2016.06.004
[13] Pedas, A., Tamme, E. (2011). Product integration for weakly singular integro-differential equations. Mathematical Modelling and Analysis, 16(1): 153-172. https://doi.org/10.3846/13926292.2011.564771
[14] Pedas, A., Tamme, E. (2006). Spline collocation method for integro-differential equations with weakly singular kernels. Journal of Computational and Applied Mathematics, 197(1): 253-269. https://doi.org/10.1016/j.cam.2005.07.035
[15] Kolk, M., Pedas, A. (2006). Numerical solution of weakly singular Volterra integro-differential equations with change of variables. In Proceedings of the 5th WSEAS International Conference on SYSTEM SCIENCE and Simulation in Engineering, Tenerife, Canary Islands, Spain, pp. 465-470. https://doi.org/10.5555/1974329.1974423
[16] Gu, Z.D. (2019). Spectral collocation method for weakly singular Volterra integro-differential equations. Applied Numerical Mathematics, 143: 263-275. https://doi.org/10.1016/j.apnum.2019.04.011
[17] Jerri, A. (1999). Introduction to Integral Equations with Applications. John Wiley & Sons.
[18] Wazwaz, A.M. (2015). A First Course in Integral Equations, World Scientific Publishing Company. http://doi.org/10.1142/9571
[19] Lakshmikantham, V. (1995). Theory of Integro-Differential Equations. CRC Press.
[20] Brunner, H. (2017). Volterra Integral Equations: An Introduction to Theory and Applications. Cambridge University Press.