Application of Laplace Transform Method for Solving Weakly-Singular Integro-Differential Equations

Rokan Khaji*, Suhad K. Hameed†

College of Sciences, University of Diyala, Diyala 32001, Iraq

Corresponding Author Email: rkhaji2@gmail.com

Copyright: ©2024 The authors. This article is published by IIETA and is licensed under the CC BY 4.0 license (http://creativecommons.org/licenses/by/4.0/).

https://doi.org/10.18280/mmep.mmepl.110428

Received: 15 November 2023
Revised: 18 January 2024
Accepted: 30 January 2024
Available online: 26 April 2024

Keywords:
Weakly-Singular, integro-differential equations, stability Laplace transform method

1. INTRODUCTION

Weakly singular integro-differential equations (WSIDE) have been growing importance enormously as it plays a key role to represent many typically in phenomena of physics and engineering as mathematical modeling. For instance, diffusion of separated particles in turbulent fluids [1], elasticity and fracture in mechanics [2], biosciences [3], potential problems, thermal conductivity problems, materials, radiative equilibrium, the Dirichlet problems [4-6] and so on.

So a variety works focused its interest to tackle the equations of this form which are usually numerical methods, including an operational method [7], Bernstein series [8], Block boundary value method [9], Partition of the interval and introduction of additional parameters [10], Smoothing transformation and spline collocation [11]. The asymptotic estimations of the solution [12], Product integration [13], collocations methods as Spline, Piecewise Polynomial, and Spectral respectively [14-16], but it is well known that the results of numerical methods have an error rate.

To address the WSIDE with different orders of derivatives without error rate in results, in this paper we propose analytic efficient and simple technique to yield exact solution which is Laplace transform method (LTM). Procedure of Laplace transform (LT) with differential equations and integral equations is changing them to polynomial equations and easily can be solved, and thus the solution of the considered equation is obtained by taking the inverse Laplace transform (LT) for it [17, 18]. Our motivation of this work is to present exact solution for many real-world problems are described as WSIDE by using efficient and simple method.

The remainder of this paper is organized as follows. In Section 2, the definition of concept WSIDE is expressed. In Section 3, we discuss the existence and uniqueness of solution for WSIDE. Section 4 discusses the various stability of solution for WSIDE. In Section 5, we introduce the concept of Laplace transform and it is followed up implementation it to solve WSIDE. In Section 6, the exact solutions of the WSIDE are obtained using the Laplace transform by solving several examples. Finally, conclusions are drawn in Section 7.

2. THE WEAKLY-SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS (WSIDE)

Consider the following standard form of nth-order linear integro-differential equation of the second kind:

\[ y^{(m)}(t) = g(t) + \int_{a}^{t} k(s,t) y(s) ds, a \leq t \leq b, \]

\[ 0 < m < \infty \]

with initial conditions \( y^{(j)}(a) = y_j, j = 0, 1, \ldots, m - 1 \), where \( y^{(m)}(t) = \frac{d^m y}{dt^m}, \ m < \infty, \ g(t), k(s,t) \) are given functions and \( y(t) \) is the unknown function, also \( k(s,t) \) is denoted the kernel of the integro equation. We usually propose that the functions \( y(t) \) and \( g(t) \) are continuous or square integrable on \([a,b]\). Furthermore, Eq. (1) is also denoted a singular integral equation if the kernel \( K(x, t) \) becomes infinite at one or more points in the domain of integration.

Motivation of present work is the desire to obtain exact analytic solution by using LTM for a linear weakly singular Volterra integro-differential equation (WSVIDE) of nth order and the second kind, where the integrand is denote as a weakly singular in the sense that its integral is continuous at the singular point, that is, its kernel \( k(s,t) = \frac{1}{(t-s)^{\alpha}} \) is singular as
3. EXISTENCE AND UNIQUENESS SOLUTION OF WSIDE

This section discusses the existence and uniqueness of solution for the WSIDE and toward meeting that, it will be beneficial considering following procedure:

Let \( y^{(m)}(t) = f(t) \), where \( f(t) \) is a continuous real valued function defined on \([a, b]\). Integrating both sides \( m \)-times from \( a \) to \( t \) and using the initial conditions, yields to:

\[
y(t) = \sum_{j=0}^{m-1} \frac{(t-a)^j}{j!} y_j + \int_{a}^{t} (t-x)^{m-1} f(x) dx
\]

(2)

Eq. (1) can be written as:

\[
f(t) = g(t) + \int_{a}^{t} k(s,t) \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \left( \int_{a}^{s} (s-x)^{m-1} f(x) dx \right) ds
\]

(3)

Suppose,

\[
W(t,s) = \frac{k(s,t)}{(m-1)!} \left( \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right)
\]

So that Eq. (3) is transformed into,

\[
f(t) = g(t) + \int_{a}^{t} W(t,s) \left( \int_{a}^{s} (s-x)^{m-1} f(x) dx \right) ds
\]

(4)

Now, having the existence of solution at our disposal, we discuss it.

**Theorem 1.** Assume that \( f \in C([a, b], R), k \in C([a, b] \times [a, b], R) \) for \( a \leq s < t \leq b \) are continuous, where \( \beta_1 \) upper bound of \( |k(t,s)|, \forall (t,s) \in [a,b]^2 \), \( \beta_2 = \frac{(t-a)^m}{m!} \) and \( \beta_3 = \left| \sum_{j=0}^{m-1} \frac{(t-a)^j}{j!} y_j \right| \). Then the WSIDE in Eq. (1) satisfies the following:

i. It has at least one continuous solution.

ii. It has a unique continuous solution.

**Proof 1.** Let \( T \) be an integral operator defined by:

\[
Tf(t) = g(t) + \int_{a}^{t} W(t,s) \left( \int_{a}^{s} (s-x)^{m-1} f(x) dx \right) ds
\]

(5)

from Eqs. (4) and (5) we have:

\[
f = Tf
\]

consider the set \( \Omega_r = \{ f \in C([a, b], R): \|f\|_\infty = \sup_{x \in [a,b]}|f(x)| \leq r \} \). The radius \( r \) is the solution of an equation \( \beta_1 (t-a) (\beta_2 r + \beta_3) + \|g\|_\infty = r \), we have to prove the operator \( T \) maps the set \( \Omega_r \) into itself, to observe that, let \( f \) be any function in \( \Omega_r \). Then:

\[
|T(f)(t)| \leq |g(t)| + \int_{a}^{t} |W(t,s)| \left( \int_{a}^{s} (s-x)^{m-1} |f(x)| dx \right) ds
\]

\[
\leq \|g(t)\|_\infty + \int_{a}^{t} \frac{r(t-a)^m}{m!} |k(t,s)| \left( \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right) ds
\]

since \( k(t,s) \) is continuous on \([a,b]^2\), therefore, \( k(t,s) \) is bounded and hence \( |k(t,s)| \leq \beta_1, \forall (t,s) \in [a,b]^2, \beta_1 > 0 \).

\[
|T(f)(t)| \leq \|g(t)\|_\infty + (t-a) \beta_3 \left( \sum_{j=0}^{m-1} \frac{(s-a)^j}{j!} y_j \right)
\]

This implies that:

\[
|T(f)(t)| \leq \|g(t)\|_\infty + \beta_1 (t-a) (r \beta_2 + \beta_3)
\]

\[
= \|g(t)\|_\infty + r - (\|g(t)\|_\infty - r)
\]

(6)

Thus, \( \|T(f)\|_\infty \leq r \). Suppose \( t_1, t_2 \in [a, b], \) and without loss of generality, let \( t_1 < t_2 \). Assume \( f \in \Omega_r \).

\[
|T(f)(t_2) - T(f)(t_1)| \leq |g(t_2) - g(t_1)| + \int_{t_1}^{t_2} |W(t_2,s) - W(t_1,s)| \left( \int_{a}^{s} (s-x)^{m-1} |f(x)| dx \right) ds
\]

(7)

But the functions \( g \) and \( W \) are continuous. So, we have:

\[
|g(t_2) - g(t_1)| \rightarrow 0 \quad |W(t_2,s) - W(t_1,s)| \rightarrow 0
\]

When \( t_2 \rightarrow t_1 \) and thus \( |T(f)(t_2) - T(f)(t_1)| \rightarrow 0 \). So, the operator \( T \) maps the set \( \Omega_r \) into itself. Next, to prove the existence of fixed point of \( T \) in \( \Omega_r \), we may apply Schauder's fixed point theorem which is equivalent to solving Eq. (1), as follows. Let \( \{f_i\}_{i=1}^{\infty} \) be a sequence with \( f_i \in \Omega_r \) and suppose \( f_i \rightarrow f \in \Omega_r \), when \( i \rightarrow \infty \). Clearly that \( \|f_i\|_\infty \leq r \), for all \( i \in N \).

\[
|T(f)(t_i) - T(f)(t)| \leq \int_{a}^{t} |W(t,s)| \left| \left( \int_{a}^{s} (s-x)^{m-1} f_i(x) dx - \int_{a}^{s} (s-x)^{m-1} f(x) dx \right) \right| ds
\]

\[
\leq \int_{a}^{t} |W(t,s)| \left( \int_{a}^{s} (s-x)^{m-1} |f_i(x) - f(x)| dx \right) ds
\]

Hence by Arzela bounded convergence theorem gives:

\[
\lim_{i \rightarrow \infty} \int_{a}^{t} |W(t,s)| \left( \int_{a}^{s} (s-x)^{m-1} |f_i(x) - f(x)| dx \right) ds = 0
\]

Thus, \( |T(f_i)(t) - T(f)(t)| \rightarrow 0 \), when \( i \rightarrow 0 \) and we can conclude the operator \( T \) is continuous. Letting \( (T(f_i)) \) be a sequence for any \( f_i \in \Omega_r \). From Eq. (6), \( \|T(f_i)\|_\infty \leq r \), for all \( i \in N \), we deduce the sequence \( (T(f_i)) \) is uniformly bounded on \([a, b] \). Moreover, from inequality (7) that for any \( \varepsilon \) there exist \( \delta \) such that \( |T(f_i)(t_2) - T(f_i)(t_1)| < \varepsilon \) provided that \( |t_2 - t_1| < \delta \) for all \( i \in N \), we conclude the sequence \( (T(f_i)) \)
is an equicontinuous. By the Ascoli-Arzelà theorem [19], there exist subsequence \( (T(f_{i_k})) \subset (T(f_i)), k \to \infty \) is convergence uniformly. Consequently, the set \( T(\Omega_r) \) is compact, which implies \( T \) is completely continuous operator. Hence by Schauder fixed point theorem [17], Eq. (4) has a fixed point in \( \Omega_r \) and implies that an integro-differential equation Eq. (1) possesses at least one solution.

**Proof 2.** Let \( f \) and \( f^* \) be any two solutions of Eq. (4), then

\[
|f(t) - f^*(t)| \leq \int_a^t |W(t,s)| \left[ \int_a^s (s-x)^{m-1} f(x) \, dx \right] \, ds \leq \beta_1 \beta_2 (t-a) \beta_3 \|f(t) - f^*(t)\|_\infty \leq (t-a) \beta_1 \beta_2 \beta_3 \|f(t) - f^*(t)\|_\infty
\]

by passing supremum with respect to \( t \in [a,b] \) for both side of inequality (8) gives:

\[
(1 - (t-a) \beta_1 \beta_2 \beta_3) \|f(t) - f^*(t)\|_\infty < 0
\]

but we have \((t-a) \beta_1 \beta_2 \beta_3 < 1\). Therefore, we must have \(\|f(t) - f^*(t)\|_\infty = 0\) and this proves that \(f(t) = f^*(t)\).

Hence the Eq. (1) has a unique solution on \([a,b]\).

4. STABILITY SOLUTION OF WSIDE

In this section we investigate the stability of the solution for the WSIDE, for this using the idea of the variation of parameters formula for linear differential systems to obtain an integral equation for the solutions of Eq. (1). For this purpose, let \( X(t) \) be a fundamental matrix solution of:

\[
y^{(m)} = Ay,
\]

so that any solution of Eq. (1) with the initial functions \( \psi \) on \([t_0, r]\) is given by:

\[
y(t, \tau, \psi) = X(t)X^{-1}(\tau) \psi + \frac{1}{(m-1)!} \int_{\tau}^t X(t)X^{-1}(s) \left( \int_a^s k(s, \tau) y(s) \, ds + f(t) \right) \, ds \tag{9}
\]

and prove stability results for the system (1). We begin with giving a suitable definition of stability, then lemma and the theorems.

**Definition 1** [19]: The solution of Eq. (1) is said to be stable if there exists \( \epsilon > 0 \) and \( t_0 \in R_+ \), given a \( \delta = \delta(t_0, \epsilon) > 0 \) such that \( \|\psi\|_{t_0} = \max_{t \leq t_0} \|\psi(s)\| < \delta \) implies \( |y(t)| < \epsilon \), \( t \geq 0 \).

**Lemma 1** [20]: (Gronwall’s inequality) Let \( y(t), x(t) \in C([a, +\infty), R_+), K > 0 \) is constant and suppose that:

\[
y(t) \leq K + \int_a^t y(s)x(s) \, ds, \quad s \in [a, +\infty)
\]

then

\[
y(t) \leq K \exp\left( \int_a^t x(s) \, ds \right), \quad s \in [a, +\infty)
\]

**Theorems 2:** Assume that:

\[
|X(t)X^{-1}(s)| \leq K, \quad 0 \leq s \leq t,
\]

\[
\int_0^t \int_0^{(t-s)^{m-1}} K(s, t) \, ds \, dt \leq M_1
\]

and

\[
\int_0^t \int_0^{(t-s)^{m-1}} f(t) \, dt \, dt \leq M_2
\]

Then the solution of Eq. (1) is uniformly stable.

**Proof:** Set \( \delta(\epsilon) < \frac{\epsilon}{K M} \) and \( \|\psi\|_{t_0} < \delta(\epsilon) \) for any \( \epsilon < 0 \). Assume that there exists \( t_0 \leq t_1 \) such that \( |y(t_1)| = \epsilon \) and \( |y(t)| = \epsilon \) on \([t_0, t_1]\). By using Eq. (9), we get:

\[
|y(t)| \leq |X(t)X^{-1}(t_0)|\|\psi(t_0)\| + \frac{1}{(m-1)!} \left[ \int_{t_0}^t |X(t)X^{-1}(s)||X(s)X^{-1}(t)| \, ds \right] dt
\]

\[
|y(t)| \leq K \delta(\epsilon) + \frac{1}{(m-1)!} \left[ \int_{t_0}^t |k(s, t)||y(s)||s\| + f(t) \right] \, ds
\]

on \([t_0, t_1]\) and define \( r(t) \equiv \max_{0 \leq s \leq t} |y(s)| \) to obtain:

\[
r(t) \leq K \delta(\epsilon) + \frac{1}{(m-1)!} \left[ \int_{t_0}^t |k(s, t)||y(s)||s\| + f(t) \right] \, ds
\]

by Lemma 1 (Gronwall’s inequality), then we get:

\[
|y(t)| \leq r(t) \leq K \delta(\epsilon)
\]

\[
exp[K \int_{t_0}^t \int_0^{(t-s)^{m-1}} |k(s, t)||s\| + f(t) \, ds] \leq K \delta(\epsilon) exp[K(M_1 + M_2)]
\]

\[
|y(t)| \leq r(t) \leq K e^{K M} \delta(\epsilon) < \epsilon \quad \text{on} \quad [t_0, t_1]
\]

where, \( M = M_1 + M_2 \).

Consequently \( |y(t_1)| \leq \epsilon \) which is a contradiction. Thus, the solution of Eq. (1) is uniformly stable, completing the proof.
**Theorems 3:** Assume that:

\[
\int_0^t |X(t)X^{-1}(s)| ds \leq L, \quad \text{for} \ t \geq 0, \quad (10)
\]

\[
\sup_{t \geq 0} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \ dt < \frac{1}{L} \quad (11)
\]

and

\[
\sup_{t \geq 0} \int_0^\infty \frac{(t-s)^{m-1}}{(m-1)!} f(t) \ dt \leq M
\]

Moreover, assume also that:

\[
\lim_{s \to 0} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} K(t,u) \ du = 0 \quad \text{for all} \ t \geq 0
\]

Then the solution of Eq. (1) is asymptotically stable.

**Proof:** We first show that stability of the solution. From Eq. (11), there exists a positive constant \( \gamma \) such that:

\[
\sup_{t \geq 0} \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \ dt < \gamma
\]

where, \( 0 < \gamma < \frac{1}{L} \)

From inequality (10), there exists a positive constant \( N \) such that:

\[
|X(t)| \leq N \quad \text{for all} \ t \geq 0
\]

For any \( \epsilon > 0 \) and \( t \geq 0 \), let \( \delta = \delta(\epsilon, t_0) < \min \left\{ \frac{1}{2\epsilon}, \frac{1}{\epsilon} \right\} \). Consider the solution of Eq. (1) such that \( |\psi(t)| < \delta \). Suppose there exists \( t_1 > t_0 \) such that \( |y(t_1)| = \epsilon, |y(t)| < \epsilon \) on \([t_0, t_1]\) and \( M \leq \frac{\epsilon}{2\epsilon} \). For all \( t \in [t_0, t_1] \), we have:

\[
|y(t)| \leq |X(t)X^{-1}(t_0)||\psi(t_0)| + \frac{1}{(m-1)!} \int_0^t |X(t)X^{-1}(s)||\psi(s)| ds + \int_0^t |f(t)| ds
\]

leads to

\[
|y(t)| \leq N|X^{-1}(t_0)| \delta + L \int_{t_0}^t \frac{(t-s)^{m-1}}{(m-1)!} \int_a^s |k(s, t)||y(s)| ds + f(t) ds \leq \frac{1}{2} \gamma L \epsilon + L \left( \gamma \epsilon + \frac{\epsilon}{2L} \right) = \epsilon
\]

Thus, \( |y(t_2)| < \epsilon \), which is a contradiction. Thus, the solution of Eq. (1) is asymptotically stable.

**Theorems 4:** Suppose that there exist positive constants \( K \geq 1, \lambda \) and \( \mu \) such that:

\[
|X(t)X^{-1}(s)| \leq Ke^{-\lambda(s-t)} \quad \text{for} \ 0 \leq s \leq t < \infty \quad (12)
\]

\[
\sup_{t \geq 0} \int_0^t e^{-\mu(s-t)} \left( \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \right) dt < \frac{\lambda}{K} \quad (13)
\]

and

\[
\sup_{t \geq 0} \int_0^\infty e^{-\mu(s-t)} \left( \frac{(t-s)^{m-1}}{(m-1)!} f(t) \right) dt \leq M
\]

Then the solution of Eq. (1) is exponentially asymptotically stable.

**Proof:** For all \( t \geq t_0 \) and \( |\psi| < \frac{1}{K} \), we have

\[
|y(t)| \leq Ke^{-\lambda(t-t_0)}|\psi(t_0)| + K \int_{t_0}^t e^{-\lambda(s-t)} \left( \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \right) ds + K \int_{t_0}^t e^{-\lambda(s-t)} f(t) ds + K \int_{t_0}^t e^{-\lambda(s-t)} f(t) ds
\]

\[
\leq K e^{\lambda t} \left( \frac{(t-t_0)^{m-1}}{(m-1)!} K(t_0,t) \right) + K \int_{t_0}^t e^{-\lambda(s-t)} \left( \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \right) ds + K \int_{t_0}^t e^{-\lambda(s-t)} f(t) ds + K \int_{t_0}^t e^{-\lambda(s-t)} f(t) ds
\]

Let denote \( e^{\mu t} |y(s)| \) by \( r(t) \), it follows that:

\[
e^{\mu t} |y(t)| \leq Ke^{\mu t} e^{-\sigma(t-t_0)}|\psi(t_0)| + (\sigma r(t)) + M \int_{t_0}^t e^{-\sigma(s-t)} ds + \int_{t_0}^t e^{-\sigma(s-t)} \left( \frac{(t-s)^{m-1}}{(m-1)!} K(s,t) \right) ds + (1 - e^{-\sigma(t-t_0)}) r(t) + M e^{-\sigma(t-t_0)}
\]

From above inequality, if \( e^{\mu t} |y(s)| < e^{\mu t} |y(t)| \) for any \( s \in [0, t] \), we see that \( r(t) = e^{\mu t} |y(t)| \), we get:

\[
|y(t)| \leq K e^{\mu t} e^{-\sigma(t-t_0)}|\psi(t_0)| + (1 - e^{-\sigma(t-t_0)}) r(t) + M e^{-\sigma(t-t_0)}
\]

1102
Therefore, \( r(t) \leq K e^{\nu t}[\psi(t_o)] \) for \( t \geq t_o \). Then \( r(t) = e^{\nu t}\{y(t)\} \) implies \( |y(t)| \leq K e^{\nu(t-t_0)}[\psi(t_o)] \) for \( t \geq t_o \).

If there exists \( s \in [0, t] \) such that \( e^{\nu t}|y(s)| > e^{\nu t}|y(t)| \), we have the following two further cases:

(a) There exists \( t_1 \in [t_0, t] \) such that \( e^{\nu t_1}|y(t_1)| = r(t) \). Then from equation (****) we have:

\[
r(t_1) = e^{\nu t_1}|y(t_1)| \leq K e^{\nu t_0} e^{-\sigma(t-t_0)}[\psi(t_o)] + (1 - e^{-\sigma(t-t_0)})r(t_1) + M e^{-\sigma(t-t_0)}
\]

Thus \( r(t_1) \leq K e^{\nu t_0}[\psi(t_o)] \) for \( t_1 \geq t_0 \). Then \( e^{\nu t}|y(t)| < r(t_1) \) implies \( |y(t)| \leq K e^{-\nu(t-t_0)}[\psi(t_o)] \) for \( t \geq t_o \).

(b) There exists \( t_2 \in [0, t_o] \) such that \( e^{\nu t_2}|y(t_2)| = r(t) \). Then \( e^{\nu t}|y(t)| < e^{\nu t_2}|y(t_2)| < e^{\nu t_2}|\psi|_{t_0} \), we have \( |y(t)| \leq |\psi|_{t_0} e^{-\nu(t-t_0)} \).

Thus from (a) and (b), the solution of Eq. (1) is exponentially stable.

5. LAPLACE TRANSFORM METHOD (LTM) AND IMPLEMENTATION

5.1 Laplace transform method (LTM)

In general, the idea of a transformation is a very important in problem solving, where the difficult problem is changed in some way into an easier problem and then solve that easier problem to obtain solution and apply it to original problem.

The Laplace transform method (LTM) aims to seek solve the differential equations and integral equations easily by changing it to polynomial, and then taking the inverse Laplace transform, which lead for the solution of intended equation.

Here we present a basic concept of LTM, for a given function \( y(s) \) with respect \( s \geq 0 \), the Laplace transform method can be described as:

\[
Y(\rho) = \mathcal{L}\{y(s)\} = \int_0^\infty e^{-\rho s} y(s)ds
\]

where, \( \rho \) is real, and \( \mathcal{L} \) denotes the Laplace transform operator. Furthermore, a vanishment \( Y(\rho) \) as \( \rho \) approaches infinity is an important and necessary condition for the existence of the Laplace transform \( Y(\rho) \). This means that:

\[
\lim_{\rho \to \infty} Y(\rho) = 0
\]

As well as, there are key properties of the Laplace transforms which are used in the proposed framework are given briefly as follows:

1. Constant multiple:

\[
\mathcal{L}\{\alpha y(s)\} = \alpha \mathcal{L}\{y(s)\}, \quad \alpha \text{ is constant.}
\]

2. Linearity property: \( \alpha, \beta \) are constant.

\[
\mathcal{L}\{\alpha y(s)\} + \mathcal{L}\{\beta g(s)\} = \alpha \mathcal{L}\{y(s)\} + \beta \mathcal{L}\{g(s)\}.
\]

3. Multiplication by \( s \):

\[
\mathcal{L}\{sy(s)\} = \frac{d}{d\rho} \mathcal{L}\{y(s)\} = -Y'(\rho)
\]

4. Laplace transforms of derivatives:

\[
\mathcal{L}\{y'(s)\} = \rho \mathcal{L}\{y(s)\} - y(0),
\]

\[
\mathcal{L}\{y''(s)\} = \rho^2 \mathcal{L}\{y(s)\} - \rho y(0) - y'(0),
\]

\[
\mathcal{L}\{y'''(s)\} = \rho^3 \mathcal{L}\{y(s)\} - \rho^2 y(0) - \rho y'(0) - y''(0),
\]

\[
\vdots
\]

\[
\mathcal{L}\{y^{(n)}(s)\} = \rho^n \mathcal{L}\{y(s)\} - \rho^{n-1} y(0) - \rho y'^{(n-2)}(0) - \cdots - \rho y^{(n-1)}(0),
\]

5. Inverse Laplace transform:

\[
Y^{-1}\{Y(\rho)\} = y(s)
\]

6. Convolution theorem for Laplace transform:

\[
\mathcal{L}\{y_1 * y_2(s)\} = \mathcal{L}\{y_1(s)\} \mathcal{L}\{y_2(s)\} = \int_0^s y_1(s-t)y_2(t)dt
\]

There are also elementary Laplace transforms, some of which can be summarized briefly in Table 1.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \mathcal{L}{y(s)} = \int_0^\infty e^{-\rho s} y(s)ds )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \frac{k}{\rho} ), ( \rho &gt; 0 )</td>
</tr>
<tr>
<td>( s )</td>
<td>( \frac{1}{\rho^2} ), ( \rho &gt; 0 )</td>
</tr>
<tr>
<td>( s^n )</td>
<td>( \frac{n!}{\rho^{n+1}} ), ( \rho &gt; 0 ), ( \text{Re } n &gt; -1 )</td>
</tr>
<tr>
<td>( e^{\alpha s} )</td>
<td>( \frac{1}{\rho - \alpha} ), ( \rho &gt; \alpha )</td>
</tr>
<tr>
<td>( \delta(s-a) )</td>
<td>( e^{-\alpha a} ), ( \alpha \geq 0 )</td>
</tr>
</tbody>
</table>

5.2 Implementation

We now describe the implementation of the LTM to solve the consider WSVIDE, we can rewrite the expression for Eq. (1). After \( k(t,s) \) was consistently set to be \( k(t,s) = \frac{1}{(t-s)^a} \) as follows:

\[
y^{(m)}(s) = f(s) + \int_0^s \frac{1}{(t-s)^a} y(t)dt \quad (15)
\]

First, taking LT for both sides of Eq. (15), leading to:

\[
\mathcal{L}\{y^{(m)}(s)\} = \mathcal{L}\{f(s)\} + \int_0^s \frac{1}{(t-s)^a} \mathcal{L}\{y(t)\}dt \quad (16)
\]

by applying the convolution property of the LTM, Eq. (16) becomes:

\[
\mathcal{L}\{y^{(m)}(s)\} = \mathcal{L}\{f(s)\} + L(s^{-a})L\{y(s)\}
\]

so that:
\[ \rho^m y'(0) - \rho^{m-1} y(0) - \cdots - \rho y^{(m-2)}(0) - y^{(m-1)}(0) = F(\rho) + \frac{\Gamma(1 - \alpha)}{\rho^{1-\alpha}} Y(\rho) \]

\[ Y(\rho) = \frac{\rho^{1-\alpha} [F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0^{(m-2)} + y_0^{(m-1)}]}{\rho^{m-1-\alpha} - \Gamma(1 - \alpha)} \]

we take the inverse LT for both sides,

\[ L^{-1}[Y(\rho)] = L^{-1}\left\{\frac{\rho^{1-\alpha} [F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0^{(m-2)} + y_0^{(m-1)}]}{\rho^{m-1-\alpha} - \Gamma(1 - \alpha)}\right\} \]

hence, we find the solution of the Eq. (2) as:

\[ y(s) = L^{-1}\left\{\frac{\rho^{1-\alpha} [F(\rho) + \rho^{m-1} y_0 + \cdots + \rho y_0^{(m-2)} + y_0^{(m-1)}]}{\rho^{m-1-\alpha} - \Gamma(1 - \alpha)}\right\} \]

6. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following linear first-order WSVID

\[ y'(s) = -2\sqrt{s} + \int_0^s \frac{1}{\sqrt{s - t}} y(t) dt \] (17)

with initial condition \( y(0) = 0 \), and exact solution \( y(s) = 1 \).

By taking LT on both sides of Eq. (17), we obtain:

\[ L[y'(s)] = L[-2\sqrt{s}] + L[s^{-1/2}] L[y(s)] \]

by convolution theorem, give:

\[ L[y'(s)] = L[-2\sqrt{s}] + s^{-1/2} L[y(s)] \]

using elementary Laplace transform

\[ \rho Y(\rho) - y(0) = \frac{\Gamma\left(\frac{3}{2}\right)}{\rho^{3/2}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{1/2}} Y(\rho) \]

substituting the given initial condition and simplistic:

\[ Y(\rho)\left(\frac{\rho^{3/2} - \sqrt{\pi}}{\rho^{1/2}}\right) = \frac{3}{\rho^{3/2}} - \frac{\sqrt{\pi}}{\rho^{3/2}} \]

solving for \( Y(\rho) \):

\[ Y(\rho) = \frac{1}{\rho^{3/2}} \]

we take the inverse LT for both sides:

\[ L^{-1}[Y(\rho)] = L^{-1}\left\{\frac{1}{\rho^{3/2}}\right\} \]

thus, the solution is \( y(s) = 1 \).

Example 2. Consider the following linear second-order WSVID:

\[ y''(s) = -\frac{36}{55} s^{11/2} + \int_0^s \frac{1}{(s-t)^{3/2}} y(t) dt \] (18)

with initial conditions \( y'(0) = 0 \), \( y(0) = 0 \), and the exact solution \( y(s) = s \). Applying the LT on both sides of Eq. (18), we get:

\[ L[y''(s)] = L\left(-\frac{36}{55} s^{11/2}\right) + L\left[\int_0^s \frac{1}{(s-t)^{3/2}} y(t) dt\right] \]

by convolution theorem, give:

\[ L[y''(s)] = L\left(-\frac{36}{55} s^{11/2}\right) + L\left[s^{-1/2}\right] L[y(t)] \]

using elementary Laplace transform, this implies:

\[ \rho^2 Y(\rho) - \rho y(0) - y'(0) = -\frac{\Gamma\left(\frac{5}{2}\right)}{\rho^{5/2}} + \frac{\Gamma\left(\frac{5}{2}\right)}{\rho^{3/2}} Y(\rho) \]

substitute the given initial conditions and simplistic:

\[ Y(\rho)\left(\frac{\rho^{5/2} - \Gamma\left(\frac{5}{2}\right)}{\rho^{3/2}}\right) = \frac{5}{\rho^{5/2}} - \frac{\Gamma\left(\frac{5}{2}\right)}{\rho^{3/2}} \]

solving for \( Y(\rho) \):

\[ Y(\rho) = \frac{1}{\rho^{3/2}} \]

we take the inverse LT for both sides:

\[ L^{-1}[Y(\rho)] = L^{-1}\left\{\frac{1}{\rho^{3/2}}\right\} \]

thus, the solution is \( y(s) = s \).

Example 3. Consider the following linear third-order WSVID:

\[ y'''(s) = -\frac{4}{3} s^{3/2} + \int_0^s \frac{1}{\sqrt{s-t}} y(t) dt \] (19)

with initial conditions \( y'''(0) = 0 \), \( y'(0) = 1 \), \( y(0) = 0 \), and the exact solution \( y(s) = s \). By taking LT on both sides of Eq. (19), we obtain:

\[ L[y'''(s)] = L\left(-\frac{4}{3} s^{3/2}\right) + L\left[s^{-1/2}\right] L[y(t)] \]

Using convolution theorem, gives:

\[ L[y'''(s)] = L\left(-\frac{4}{3} s^{3/2}\right) + L\left[s^{-1/2}\right] L[y(t)] \]

which becomes:

\[ \rho^3 Y(\rho) - \rho^2 y(0) - \rho y'(0) - y''(0) = -\frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{3/2}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{1/2}} Y(\rho) \]
substitute the given initial conditions and simplistic:
\[ Y(\rho)\left(\frac{\rho^2 - \sqrt{\pi}}{\rho^2}\right) = \frac{\rho^2 - \sqrt{\pi}}{\rho^2} \]
solving for \( Y(\rho) \):
\[ Y(\rho) = \frac{1}{\rho^2} \]
we take the inverse LT for both sides:
\[ L^{-1}\{Y(\rho)\} = L^{-1}(\frac{1}{\rho^2}) \]
thus, the solution is \( y(s) = s^3 \).

**Example 4.** Consider the following linear fourth-order WSVIDE
\[ y^{(iv)}(s) = -\frac{128}{45} \frac{g}{s^4} + \int_{0}^{s} \frac{1}{(s-t)^{\frac{3}{2}}} y(t) dt \]  
(20)
with initial conditions \( y^{(iv)}(0) = 0 \), \( y'''(0) = 2 \), \( y''(0) = 0 \), \( y(0) = 0 \), and the exact solution \( y(s) = s^2 \). By taking LT on both sides of Eq. (20), we get:
\[ L\{y^{(iv)}(s)\} = L\{-\frac{128}{45} \frac{g}{s^4}\} + L\{\frac{1}{(s-t)^{\frac{3}{2}}} y(t) dt\} \]
then using convolution theorem leads us to conclude that:
\[ L\{y^{(iv)}(s)\} = L\{-\frac{128}{45} \frac{g}{s^4}\} + L\{\frac{1}{s^{\frac{3}{2}}}\} L\{y(t)\} \]
this simplifies to:
\[ \rho^4 Y(\rho) - \rho^3 y(0) - \rho^2 y'(0) - \rho y''(0) - y'''(0) \]
\[ = -\frac{2\Gamma\left(\frac{1}{4}\right)}{\rho^{\frac{13}{4}}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\rho^{\frac{5}{4}}} Y(\rho) \]
substitute the given initial conditions and simplistic:
\[ Y(\rho)\left(\frac{\rho^{\frac{17}{4}} - \Gamma\left(\frac{1}{4}\right)}{\rho^{\frac{13}{4}}}\right) = \frac{2(\rho^{17/4} - \Gamma\left(\frac{1}{4}\right))}{\rho^{\frac{15}{4}}} \]
solving for \( Y(\rho) \):
\[ Y(\rho) = \frac{2}{\rho^3} \]
we take the inverse LT for both sides:
\[ L^{-1}\{Y(\rho)\} = L^{-1}(\frac{2}{\rho^3}) \]
thus, the solution is \( y(s) = s^3 \).

7. CONCLUSIONS

In this work, we have proposed a study for the weakly singular integro-differential equations by introducing a theory framework that has been achieved by proving the theorems of an existence, uniqueness, and stabilities of solution. Along with that, we have presented a simple and powerful tool to solve WSIDE which is the Laplace transform technique that was investigated to solve WSIDE for different orders. We have succeeded to obtain the exact solutions by testing our proposed method on different examples of different orders. Encouragingly for next work, the proposed method will be used to solve the system of WSIDE.

REFERENCES


