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Generalized Hyers-Ulam-Rassias Stability of an Euler-Lagrange Type Cubic Functional Equation in Non-Archimedean Quasi-Banach Spaces

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ABSTRACT

In this study, we aim to prove the generalized Hyers-Ulam-Rassias ($_G$HUR) stability for the following Euler-Lagrange (EL) type cubic functional equation $f(\mathrm{b}x+y)+f(x+by)=(\mathrm{b}+1)(\mathrm{b}-1)^2[f(x)+f(y)]+b(\mathrm{b}+1)f(x+y)$in non-Archimedean (n.A) quasi-Banach spaces and n.A (n,β) Banach spaces. In recent decades, the stability of functional equations has emerged as one of the most intriguing and engaging topics, as it leads to the applications of functional equations in various domains such as algebraic geometry, Group theory, Mechanics etc., This study is to investigate the $_G$HUR stability for the above equation using Hyers direct method. Furthermore, we obtain the stability results for the aforementioned equation with an illustrative example for the n.A case. With the study of the example one may easily understand how the stability result of functional equations in n.A case differs from the setting of classical Banach spaces.

1. INTRODUCTION

Functional equations are mandatory for the examination of stability problems in a wide range of contexts. During a gathering of the mathematics committee at the University of Wisconsin in 1940, Ulam [1] was the first person to bring up the idea of the stability problem within the context of a functional equation. Further, he addressed many unanswered questions among the topics in the same year. The following is an essential question within the domain of functional equations theory: When $G_1$is a group and G_2 is a metric group and for a given $\epsilon>0$ does there exists a $\delta>0$ such that $d(h(xy),h(x)h(y))<\delta$ for a function $h: G_1\rightarrow G_2$ and for all $x,y\in G_1$, implies that there exists a homomorphism $H: G_1\rightarrow G_2$ satisfying the inequality $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? This understanding of stability is crucial to guaranteeing that mathematical models and systems are reliable and predictable in real-world applications. Hyers [2] provided the first partial response to the question of Ulam in 1941 by taking Banach spaces into consideration. This has helped to develop a better understanding of the features of stability in mathematical systems. The answer given by Hyers was extended for additive mappings by Aoki [3] in 1950. Later, Rassias [4] gave the generalized version of Hyer's theorem for linear mappings in 1978. For the previous thirty years, the generalization of HU stability has been greatly influenced by the Rassias result. This was demonstrated by Gajda [5] in 1991 for the case of p>1. Both Rassias and Semrl demonstrated that the aforementioned statement cannot be made for p=1. In 1994, Gavruta [6] obtained the generalization of HU stability for approximately additive mappings. Kenary [7] discussed a cubic functional equation and its stability over random normed spaces. The $_G$HUR stability of the same over Banach spaces has been discussed by Jun and Kim [8]. In 2006, Jun and Kim [9] determined the stability results for Jensen-type functional equations of cubic mappings. Ramachandran and Sangeetha [10] obtained the stability results of generalized quadratic functional equation over n.A normed spaces. In 2009, Moradlou et al. [11] achieved the stability concerning homomorphisms and derivation on C^*-ternary rings in connection with an EL type additive mapping. The stability results for an Euler-Lagrange-Rassias (ELR) type functional equation was obtained by Pietrzyk [12] in 2006. In Banach spaces, the $_G$HUR stability for an EL type cubic functional equation was carried out by Najati and Moradlou [13] in the year 2009. In 2011, Eskandani et al. [14] demonstrated the stability of a generalized mixed type of a functional equation in quasi-β-normed spaces. Gordji et al. [15] discussed HUR stability for the additive-quadratic type of the equations over fuzzy Banach spaces in the year 2012. Koh and Kang [16] examined an ELR type quartic functional equation for the solution and also for its stability in various quasi normed spaces in 2013. The concept of p-Adic numbers were mainly initiated by Hensel [17] in the year 1897. The involvement of p-Adic numbers in analyzing the stability of functional equations is complex and significant. Moreover, the utilization of p-Adic methods extends to fields like number theory, algebraic geometry, and cryptography, enriching the exploration of functional equation stability. In stability theory, n.A spaces are useful for examining how systems behave under perturbations. Arriola and Beyer [18] established the stability in the setting of n.A spaces for the first time. For more understanding of n.A and stability of functional equations in various spaces one may refer to studies [19-26]. In the year 2011, Eskandani and Gavruta [27] obtained the stability results of the pexiderized Cauchy functional equation in n.A spaces. In the same year, perturbation of higher ring derivations in n.A Banach algebras was discussed by Eshaghi Gordji et al. [28] using fixed-point method. In 2011, Park [29] discussed various stability results considering different mappings in 2-Banach spaces and related topics. Also, the Ulam stability in 2-Banach spaces uses fixed point approach and this was demonstrated by Ciepliński [30] in 2021. The idea of 2-Banach spaces extends the conventional concept of Banach spaces by including a dual-norm structure, leading to a deeper understanding of vector spaces and their characteristics. Wang et al. [31] investigated the stability of additive ρ-type functional inequalities in the setting of n.A 2-normed spaces in 2021. Aribou and Kabbaj [32] explored the stability of N-dimensional quadratic type functional inequality in n.A Banach spaces. In the same year, Wang [33] discussed the same for a mixed type quadratic and cubic functional equation in n.A (n, β) normed spaces. HU stability of functional equation deducing from quadratic mapping in n.A (n, β) normed spaces was proved by Alessa et al. [34] in 2021. In 2015, Yang et al. [35] illustrated the stability of diverse types of functional equation within (n, β) normed spaces. The (n, β) normed space stands out as an exceptionally adaptable and enlightening framework, providing a nuanced view of the convergence and stability of mathematical systems. In 2023, Ramakrishnan and Uma [36] established the stability for quadratic-additive type functional equation in n.A quasi Banach spaces. Very recently, Ramakrishnan et al. [37] obtained the $_G$HU stability of a Bi-Quadratic mapping for the case of n.A spaces. Jun and Kim [38] investigated the $_G$HUR stability for an EL type cubic functional equation in quasi-Banach spaces. In the year 2022, more improved results of those equations were obtained by Dung and Sintunavarat [39].

Our study focuses mainly to obtain the $_G$HUR stability of following Eq. (1) in n.A quasi-Banach spaces and n.A (n,β) Banach spaces.

$$\begin{aligned} f(\mathrm{b}x+y)+f(x+by) &= (\mathrm{b}+1)(\mathrm{b}-1)^2[f(x)+f(y)] \\ &\quad + b(\mathrm{b}+1)f(x+y) \end{aligned} \tag{1}$$

It is obvious that the solution of Eq. (1) is $f(x) = cx^3$. The solution is called a cubic function. This research article is organized as follows: In section 2, we will present some preliminary ideas relevant to our topic which will help us to obtain the results for our main investigation. In section 3 we will explore the $_G$HUR stability of Eq. (1) in n.A quasi-Banach spaces, n.A p-Banach spaces and n.A (n,β) Banach spaces by Hyers direct method, which is provided with an example. In section 4 conclusion of this study will be summarized.

2. PRELIMINARIES

This section offers fundamental results regarding different norms within a linear space.

Definition 2.1 [24] The function or valuation $|.|: \mathbb{K} \to \mathbb{R}$ is said to be non-Archimedean over the field $\mathbb{K}$ if it meets the three requirements listed below, where $x, y \in \mathbb{K}$:

(i) $|x| \geq 0$ and $|x| = 0$ if and $only$ if $x = 0$;
(ii) $|xy| = |x|.|y|$;
(iii) $|x+y| \leq max\{|x|, |y|\}$.

In the above conditions, (iii) is called a stronger triangle inequality. The field $\mathbb{K}$, with the above valuation is called n.A field. The valuation is called trivial if $|x| = 0$ when $x = 0$ and $|x| = 1$, when $x \neq 0$. Otherwise, it is called non-trivial.

In this study we consider $\mathbb{K}$ as a complete non-trivially valued n.A field.

The best example for a n.A field is the field of p-Adic numbers. Also, in a n.A field, any triangle is isosceles and any two spheres are either identical or disjoint. Every point of the sphere is its centre.

Definition 2.2 [24] Let the space X be considered as a linear space over $\mathbb{K}$. A function or norm $\|.\|: X \to \mathbb{R}$ is called n.A if it meets the below three conditions.

(i) $\| x \| = 0$ iff $x = 0$;
(ii) $\| rx \| = |r| \| x \|$ for all $x \in X, r \in \mathbb{K}$;
(iii) $\| x+y \| \leq max\{\| x \|, \| y \|\}$ for all $x, y \in X$.

the space X with the above norm is called a n.A normed space. From (iii), we have $\| x_n - x_m \| \leq max\{\| x_{j+1} - x_j \|; m \leq j \leq n-1\}$ $(n \geq m)$.

Definition 2.3 [24] In a n.A normed space X, A sequence $\{x_n\}$ is termed as Cauchy iff $\{x_{n+1} - x_n\}$ converges to "zero". Further the space is called complete if every Cauchy sequence converges in it.

Definition 2.4 Let the space X be considered as a linear space over $\mathbb{K}$. Then a norm $\|.\|: X \to \mathbb{R}$ is known as n.A quasi norm if it satisfies the conditions outlined below:

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$;
(ii) $\| \lambda x \| = |\lambda| \| x \|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
(iii) There is a constant $\kappa \geq 1$ such that

$$\| x+y \| \leq \kappa\, max(\| x \|, \| y \|) \tag{2}$$

where $x, y \in X$. The space X with the above norm is called a n.A quasi normed space. Also, it is important to note that every n.A normed space is a n.A quasi normed space.

Definition 2.5 A complete n.A quasi normed space is a n.A quasi-Banach space. A n.A quasi norm $\|.\|$ is said to be a p-norm if the third condition is replaced by $\| x+y \|^p \leq \max\{\| x \|^p, \| y \|^p\}$ for every $x, y \in X$ and $0 < p \leq 1$. This space becomes a Banach space when it is complete.

Definition 2.6 [34] Let the space X be considered as a linear space over $\mathbb{K}$. Then for a positive integer n and for a constant β with $0 < \beta \leq 1$, the function $\|., \dots, .\|_\beta: X^n \to \mathbb{R}$ is said to be (n, β) norm on X if

(i) $\|x_1, x_2, \dots, x_n\|_\beta = 0$ iff $x_1, x_2, \dots, x_n$ are linearly dependent;
(ii) $\|x_1, x_2, \dots, x_n\|_\beta$ is invariant under permutation of $x_1, x_2, \dots, x_n$;
(iii) $\|\alpha x_1, x_2, \dots, x_n\|_\beta = |\alpha|^\beta \|x_1, x_2, \dots, x_n\|_\beta$;

(iv) $\|x+y, x_2, \dots, x_n\|_\beta \leq \max\{\|x, x_2, \dots, x_n\|_\beta, \|y, x_2, \dots, x_n\|_\beta\}$.

where $x, y, x_1, x_2, \dots, x_n \in X$ and $\alpha \in \mathbb{K}$.

Then $(X, \|., \dots, .\|_\beta)$ is termed as a n.A (n,β)-normed space.

3. STABILITY RESULTS OF EQUATION (1) OVER NON-ARCHIMEDEAN QUASI-BANACH SPACES AND NON-ARCHIMEDEAN (n,β)-BANACH SPACES

Theorem 3.1 Let $\phi: X \times X \to \mathbb{R}^+$ for which $f: X \to Y$ (X denotes a n.A quasi normed space, while Y represents a n.A quasi Banach space) satisfies the inequality:

$$\| f(bx+y) + f(x+by) - (b+1)(b-1)^2[f(x)+f(y)] - b(b+1)f(x+y) \| \leq \phi(x,y) \tag{3}$$

and

$$\lim_{i\to\infty} \left(\frac{\kappa}{|b|^3}\right)^i \phi(b^i x, b^i y) \tag{4}$$

converges, where $x, y \in X$. Then an EL type cubic mapping exists uniquely as $T: X \to Y$ fulfills the Eq. (1) and

$$\|f(x) - T(x)\| \leq \frac{\kappa}{|b|^3}\left[\max\left\{\left(\frac{\kappa}{|b|^3}\right)^i \phi(b^i x, 0)\right\}\right] : 0 \leq i < n \tag{5}$$

where, T is given by:

$$T(x) = \lim_{n\to\infty} \frac{f(b^n x)}{b^{3n}} \tag{6}$$

for $x \in X$.

Proof. To demonstrate the stability results, it is necessary to establish the following:

(i) The sequence $\left\{\frac{f(b^n x)}{b^{3n}}\right\}$ is a Cauchy sequence.
(ii) If $T(x) = \lim_{n\to\infty} \frac{f(b^n x)}{b^{3n}}$ then T is cubic.
(iii) Further T satisfies $\| S(x) - T(x) \| \leq \delta$.
(iv) T is unique.

For, first we substitute $y = 0$ in inequality (3) and divide the inequality with $|b|^3$, then, we obtain:

$$\left\|\frac{f(bx)}{b^3} - f(x) - \frac{(b+1)(b-1)^2 f(0)}{b^3}\right\| \leq \frac{1}{|b|^3}\phi(x,0)$$

now, let $f(0) = 0$ then it may be written as:

$$\left\|f(x) - \frac{f(bx)}{b^3}\right\| \leq \frac{1}{|b|^3}\phi(x,0) \tag{7}$$

now changing x by $b^i x$ in inequality (7) and dividing either side by $|b|^{3i}$, we get the following:

$$\left\|\frac{f(b^i x)}{b^{3i}} - \frac{f(b^{i+1}x)}{b^{3(i+1)}}\right\| \leq \frac{1}{|b|^3}\frac{\phi(b^i x, 0)}{|b|^{3i}} \tag{8}$$

by using the method of mathematical induction on n and taking summation from $i = 0,1,\dots,n-1$ we get:

$$\left\|f(x) - \frac{f(b^n x)}{b^{3n}}\right\| \leq \frac{\kappa}{|b|^3}[\max\{(\frac{\kappa}{b^3})^i \phi(b^i x, 0), (\frac{\kappa}{b^3})^{n-1}\phi(b^{n-1}x, 0)\}] : 0 \leq i \leq n-2 \tag{9}$$

We obtain the inequality (7) when $n = 1$. Now we take $n = n+1$ in inequality (9), then we have:

$$\begin{aligned}\left\|f(x) - \frac{f(b^{n+1}x)}{b^{3(n+1)}}\right\| &\leq \max\{\kappa\left\|f(x) - \frac{f(bx)}{b^3}\right\|, \kappa\left\|\frac{f(bx)}{b^3} - \frac{f(b^{n+1}x)}{b^{3(n+1)}}\right\|\} \leq \max\{\frac{\kappa}{|b|^3}\phi(x,0), \\ &\frac{\kappa^2}{|b|^3}\frac{\phi(bx,0)}{b^3}, \dots, \frac{\kappa^{n+1}}{|b|^3}\frac{\phi(b^n x, 0)}{(b^3)^n}\} \\ &\leq \frac{\kappa}{|b|^3}[\max\{(\frac{\kappa}{b^3})^i\phi(b^i x, 0), \\ &(\frac{\kappa}{b^3})^n \phi(b^n x, 0)\}] : 0 \leq i \leq n-1\end{aligned}$$

now we can prove the sequence $\{\frac{f(b^n x)}{b^{3n}}\}$ is convergent. For, we can divide the inequality (9) by $|b|^{3m}$ and also change x by $b^m x$, hence, we can see the inequality (9) as follows:

$$\begin{aligned}&\left\|\frac{f(b^{n+m}x)}{b^{3(n+m)}} - \frac{f(b^m x)}{b^{3m}}\right\| \\ &= \frac{1}{|b|^{3m}}\left\|\frac{f(b^n b^m x)}{b^{3n}} - f(b^m x)\right\| \\ &\leq \frac{\kappa}{|b|^{3m+3}}[\max\{(\frac{\kappa}{b^3})^i \phi(b^i b^m x, 0), \\ &(\frac{\kappa}{b^3})^{n-1}\phi(b^{n-1}b^m x, 0)\}] \\ &\leq \frac{\kappa}{\kappa^m |b|^3}[\max\{(\frac{\kappa}{b^3})^{m+i}\phi(b^{m+i}x, 0), \\ &(\frac{\kappa}{b^3})^{m+n-1}\phi(b^{m+n-1}x, 0)\}], \quad n, m > 0\end{aligned} \tag{10}$$

which converges to 0 as $m \to \infty$, hence we obtain the sequence $\{\frac{f(b^n x)}{b^{3n}}\}$ as Cauchy. So, we can establish:

$$T(x) = \lim_{n\to\infty}\frac{f(b^n x)}{b^{3n}}$$

letting $n \to \infty$ in inequality (9), we achieve the inequality (5). Now we prove that T satisfies Eq. (1). For, we can replace x, y as $b^n x, b^n y$ respectively in inequality (3) and divide the equation by $|b|^{3n}$. Then we get:

$$\begin{aligned}&(\frac{1}{b^3})^n \| f(b.b^n x + b^n y) + f(b^n x + b.b^n y) \\ &-(b+1)(b-1)^2[f(b^n x) + f(b^n y)] \\ &-b(b+1)f(b^n x + b^n y) \| \\ &\leq (\frac{\kappa}{|b|^3})^n \phi(b^n x, b^n y)\end{aligned}$$

then by letting $n \to \infty$, we obtain the result. Suppose we assume that another function $S: X \to Y$ meets the Eq. (1) and also the inequality (5). Let $x = b^n x$. Then we have $S(b^n x) = b^{3n}S(x)$, now from inequality (5) we have:

$$\begin{aligned}\| S(x)-T(x) \| &= |b|^{-3n} \| S(b^n x)-T(b^n x) \| \\ &\le |b|^{-3n}(\| S(b^n x)-f(b^n x)-c \| \\ &\quad +\| f(b^n x)+c-T(b^n x) \|) \\ &\le |b|^{-3n}\max\{\kappa \| S(b^n x)-f(b^n x)-c \|, \\ &\quad \kappa \| f(b^n x)+c-T(b^n x) \|\} \\ &\le \frac{\kappa}{|b|^3} max\{(\frac{\kappa}{|b|^3})^{n+i}\phi(b^{n+i}x,0)\} : 0\le i<n\end{aligned}$$

further letting $n\to\infty$, we obtain the result.

Corollary 3.2 Let $|3|<1$ and let $\tau:[0,\infty)\to[0,\infty)$ be defined by:

$$\tau(s)=\begin{cases}\frac{|27|^n}{n+1}, s=|3|^n r, n\in\mathbb{N}\cup\{0\}, r>0\\ s \qquad \text{otherwise}\end{cases}$$

suppose that $\delta>0$, and if X is a quasi-normed space and Y is a quasi-Banach space, $f:X\to Y$ satisfies the inequality (3) when b= 3, that is,

$$\begin{aligned}\|f(3x+y)+f(x+3y)-12f(x+y)-16f(x) \\ -16f(y)\| \le \delta\, max\{\tau\|x\|,\tau\|y\|\}\end{aligned}$$

for all $x,y\ in\ X$. Then another mapping exists uniquely as $T:X\to Y$ satisfies the Eq. (1) such that:

$$\|f(x)-T(x)\| \le \frac{1}{|81|}\delta\tau\|x\|$$

Proof. Define $\phi:X\times X\to[0,\infty)$ by $\phi(x,y)=\delta\, max\{\tau\|x\|,\tau\|y\|\}$ then we have:

$$\begin{aligned}&\lim_{n\to\infty}\frac{\kappa^n}{|3|^{3n}}\phi(3^n x,3^n y)\\ &\le \lim_{n\to\infty}\frac{\kappa^n}{|3|^{3n}}\delta\, max\{\tau\|3^n x\|,\tau\|3^n y\|\}\\ &\le \lim_{n\to\infty}\frac{\kappa^n}{|3|^{3n}}\delta\, max\left\{\frac{|3|^{3n}}{n+1}\tau\|x\|,\frac{|3|^{3n}}{n+1}\tau\|y\|\right\}\\ &\le \lim_{n\to\infty}\frac{\kappa^n}{|3|^{3n}}\frac{|3|^{3n}}{n+1}\delta\, max\{\tau\|x\|,\tau\|y\|\}\\ &\le \lim_{n\to\infty}\delta\frac{\kappa^n}{n+1}max\{\tau\|x\|,\tau\|y\|\}\\ &\le \lim_{n\to\infty}\delta\frac{\kappa^n}{n+1}\ \phi(x,y)=0\end{aligned}$$

further let $\bar{\phi}(x) = max\left\{\frac{\kappa^n}{|3|^{3n}}\ \phi(3^n,0)\right\}$

$$\begin{aligned}&= max\left\{\frac{\kappa^n}{|3|^{3n}}\frac{|3|^{3n}}{n+1}\phi(x,0)\right\}\\ &= max\left\{\frac{\kappa^n}{n+1}\phi(x,0)\right\}=\frac{\kappa^n}{n+1}\phi(x,0)\end{aligned}$$

using Theorem 3.1, which yields the required result.

Theorem 3.3 Let a mapping $\phi:X^2\to\mathbb{R}^+$ for which $f:X\to Y$ (X denotes a n.A quasi normed space, while Y represents a n.A quasi Banach space) satisfies

$$\begin{aligned}\| f(\mathrm{b}x+y)+f(x+\mathrm{b}y)-(\mathrm{b}+1)(\mathrm{b}-1)^2[f(x) \\ +f(y)]-b(\mathrm{b}+1)f(x+y) \| \\ \le \phi(x,y)\end{aligned} \tag{11}$$

and

$$\lim_{i\to\infty}(|\mathrm{b}|^3\kappa)^i\phi(\frac{x}{b^i},\frac{y}{b^i}) \tag{12}$$

converges for each $x,y\in X$. Then another EL type cubic mapping exists uniquely as $T:X\to Y$ fulfills the Eq. (1) and:

$$\| f(x)-T(x) \| \le \frac{1}{|\mathrm{b}|^3}\max\,(|\mathrm{b}|^3\kappa)^i\phi(\frac{x}{b^i},0): 0\le i<n \tag{13}$$

where, T is given by:

$$T(x)=\lim_{n\to\infty}b^{3n}f(\frac{x}{b^n}) \tag{14}$$

for $x\in X$.

Proof. Replace x by $\frac{x}{b^i}$ in inequality (7) and multiply both sides of inequality (7) by $|\mathrm{b}|^{3i}$ we get:

$$\left\|b^{3i}f(\frac{x}{b^i})-b^{3(i+1)}f(\frac{x}{b^{i+1}})\right\| \le \frac{1}{|\mathrm{b}|^3}|\mathrm{b}|^{3i}\phi(\frac{x}{b^i},0) \tag{15}$$

using an induction method on n and taking summation from $i=0,1,2,\ldots,n$ in inequality (15) we obtain that:

$$\begin{aligned}\left\|f(x)-b^{3n}f\left(\frac{x}{b^n}\right)\right\| \le \frac{1}{|\mathrm{b}|^3}\max\{(|\mathrm{b}|^3\kappa)^i\phi(\frac{x}{b^i},0),\\ (|\mathrm{b}|^3\kappa)^{n-1}\phi(\frac{x}{b^{n-1}},0)\}: 0\le i\le n-2\end{aligned} \tag{16}$$

now, we take $n=n+1$ in inequality (16), then

$$\begin{aligned}&\left\|f(x)-b^{3(n+1)}f\left(\frac{x}{b^{n+1}}\right)\right\| \le \max\{\kappa\left\|f(x)-b^3f\left(\frac{x}{b}\right)\right\|,\\ &\kappa\left\|b^3f(\frac{x}{b})-b^{3(n+1)}f(\frac{x}{b^{n+1}})\right\|\}\\ &\le \frac{1}{|b|^3}\max\{(|b|^3\kappa)^i\phi\left(\frac{x}{b^i},0\right),(|b|^3\kappa)^n\phi(\frac{x}{b^n},0)\}\\ &: 0\le i\le n-1\end{aligned}$$

let us now show that $\{b^{3n}f(\frac{x}{b^n})\}$ is convergent. For, multiplying the inequality (16) by $|b|^{3m}$ and also changing x by $b^m x$, we get

$$\begin{aligned}&\left\|b^{3m}f(b^m x)-b^{3(m+n)}f(\frac{b^m x}{b^n})\right\|\\ &= |b|^{3m}\left\|f(b^m x)-b^{3n}f(\frac{b^m x}{b^n})\right\|\\ &\le \frac{|b|^{3m}}{|b|^3}max\{(|b|^3\kappa)^i\phi(\frac{b^m x}{b^i},0),\\ &(|b|^3\kappa)^{n-1}\phi(\frac{b^m x}{b^{n-1}},0) \le \frac{|b|^{3m}}{\kappa^m|b|^3}\max\{(|b|^3\kappa)^{m+i}\\ &\phi\left(\frac{b^m x}{b^i},0\right),(|b|^3\kappa)^{m+n-1}\phi(\frac{b^m x}{b^{n-1}},0)\},: 0\le i\le n-1\\ &n,m>0\end{aligned}$$

which converges to 0 as $m\to\infty$. Hence the sequence $\{b^{3n}f(\frac{x}{b^n})\}$ is Cauchy. Therefore, we may define:

$$T(x)=\lim_{n\to\infty}b^{3n}f(\frac{x}{b^n})$$

where $x \in X$. Then by letting $n \to \infty$ in inequality (16), we obtain the result. The uniqueness of T can be obtained in a similar approach given to theorem 3.1.

Theorem 3.4 Let X be a n.A quasi-normed space and Y *be* a n.A p-Banach space. Let $\phi: X \times X \to \mathbb{R}^+$ and $f: X \to Y$ satisfies the functional inequality:

$$\begin{aligned}\| f(\mathrm{b}x+y)+f(x+by)-(\mathrm{b}+1)(\mathrm{b}-1)^2[f(x) \\ +f(y)]-b(\mathrm{b}+1)f(x+y)\| \\ \leq \phi(x,y)\end{aligned} \tag{17}$$

and

$$\lim_{i\to\infty}\frac{\phi(b^i x, b^i y)^p}{|\mathrm{b}|^{3pi}} \tag{18}$$

converges. Then another EL type cubic function exists uniquely as $T: X \to Y$ given by Eq. (6), which holds Eq. (1) and:

$$\begin{aligned}\|f(x)-T(x)\| \leq \frac{1}{|\mathrm{b}|^3}[\max\{\frac{\phi(b^i x,0)^p}{|\mathrm{b}|^{3pi}}\}]^{\frac{1}{p}} \\ : l \leq i < m\end{aligned} \tag{19}$$

Proof. From the inequality (8) and by the definition of p-norm, we have:

$$\begin{aligned}\left\|\frac{f(b^l x)}{b^{3l}}-\frac{f(b^m x)}{b^{3m}}\right\|^p \leq \left\|\frac{f(b^i x)}{b^{3i}}-\frac{f(b^{i+1}x)}{b^{3(i+1)}}\right\|^p \\ : l \leq i \leq m-1 \\ \leq \frac{1}{|\mathrm{b}|^3} max\left\{\frac{\phi(b^i x,0)^p}{|\mathrm{b}|^{3pi}}\right\}\end{aligned} \tag{20}$$

for each x belongs to X and all l, m with $m > l \geq 0$. Letting $m \to \infty$ it tends to zero and we get the sequence $\{\frac{f(b^m x)}{b^{3m}}\}$ is Cauchy for every $x \in X$ also, by the completeness of Y we can write:

$$\mathrm{T}(x) = \lim_{m\to\infty}\frac{f(b^m x)}{b^{3m}}$$

now we prove that the function T fulfills the Eq. (1). For, we put $x = b^m x$ and $y = b^m y$ in inequality (17) and divide either side by $|b|^{3mp}$ and taking p-norm we obtain:

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{|b|^{3mp}}\| f\big(b^m(bx+y)\big)+f\big(b^m(x+by)\big) \\ -(b+1)(b-1)^2[f(b^m x)+f(b^m y)] \\ -b(b+1)f(b^m(x+y))\|^p \leq \lim_{m\to\infty}\frac{\phi(b^m x, b^m y)^p}{|b|^{3mp}}\end{aligned}$$

then letting $m \to \infty$, we obtain the result. Suppose that another cubic mapping $S: X \to Y$ exists, which fulfills Eq. (1) and inequality (19). Then, we have $S(b^m x) = b^{3m}S(x)$, now from inequality (19) we have

$$\begin{aligned}\| S(x)-T(x)\|^p = |b|^{-3mp}\| S(b^m x)-T(b^m x)\|^p \\ \leq |b|^{-3mp}(\| S(b^m x)-f(b^m x)-c \\ +f(b^m x)+c-T(b^m x)\|^p) \\ \leq |\mathrm{b}|^{-3mp}\max\{\| S(b^m x)-f(b^m x)-c\|^p, \\ \| f(b^m x)+c-T(b^m x)\|^p\} \\ \leq \frac{1}{|b|^{3p}} max\left\{\frac{\phi(b^{m+i}x,0)^p}{|b|^{3(m+i)p}}\right\} : 0 \leq i < n\end{aligned}$$

hence by letting $n \to \infty$ we obtain the proof of the theorem.

Theorem 3.5 Let X be a n.A quasi-normed space and Y be a n.A p-Banach space. Let $\phi: X \times X \to \mathbb{R}^+$ and f from X to Y satisfies the functional inequality:

$$\begin{aligned}\| f(\mathrm{b}x+y)+f(x+by)-(\mathrm{b}+1)(\mathrm{b}-1)^2[f(x) \\ + f(y)]-b(\mathrm{b}+1)f(x+y)\| \\ \leq \phi(x,y)\end{aligned} \tag{21}$$

and

$$\lim_{i\to\infty}|\mathrm{b}|^{3pi}\phi(\frac{x}{b^i},\frac{y}{b^i})^p \tag{22}$$

converges. Then another EL type cubic mapping exists uniquely as $T: X \to Y$ defined by inequality (14) which holds the Eq. (1) and

$$\begin{aligned}\| f(x)-T(x)\| \leq \frac{1}{|\mathrm{b}|^3}[\max\{(|\mathrm{b}|^{3pi})\phi(\frac{x}{b^i},0)^p\}]^{\frac{1}{p}} \\ : l \leq i < m\end{aligned} \tag{23}$$

Proof. From the inequality (15) and by the definition of p-norm, we have:

$$\begin{aligned}\left\|b^{3l}f\left(\frac{x}{b^l}\right)-b^{3m}f\left(\frac{x}{b^m}\right)\right\|^p \\ \leq \left\|b^{3i}f\left(\frac{x}{b^i}\right)-b^{3(i+1)}f\left(\frac{x}{b^{i+1}}\right)\right\|^p \\ : l \leq i \leq m-1 \\ \leq \frac{1}{|\mathrm{b}|^3} max\{|\mathrm{b}|^{3pi}\phi\left(\frac{x}{b^i},0\right)^p\}\end{aligned} \tag{24}$$

where $x\ in\ X$. Hence by letting $m \to \infty$, the sequence $\{b^{3m}f(\frac{x}{b^m})\}$ is said to be Cauchy and it converges since Y is complete. Hence, we can write:

$$T(x) = \lim_{m\to\infty} b^{3m}f(\frac{x}{b^m})$$

where $x \in X$. The remaining proof follows a similar approach to the preceding theorem.

Theorem 3.6 Let $\phi: X^{n+1} \to \mathbb{R}^+$ be a function with:

$$\lim_{i\to\infty}\phi\left(\frac{b^i x, b^i y, v_2, \dots, v_n}{|b|^{i\beta}}\right) = 0 \tag{25}$$

for $x, y, v_2, \dots, v_n$ in X. Presume that $f: X \to Y$ (where X is a n.A (n,β)- normed space and Y is a n.A (n,β)- Banach space) is a mapping that meets the inequality:

$$\left\|D_f(x,y), v_2, \dots, v_n\right\|_\beta \leq \phi(x,y,v_2,\dots,v_n) \tag{26}$$

where, $D_f(x,y)$ is defined as: $f(bx+y)+f(x+by)-(b+1)(b-1)^2[f(x)+f(y)]-b(b+1)f(x+y)$ and where $x, y, v_2, \dots, v_n \in X$. Then an another EL cubic type mapping exists uniquely as $T: X \to Y$ such that

$$\begin{aligned}\|f(x)-T(x), v_2, \dots, v_n\|_\beta \\ \leq \lim_{i\to\infty}\max\{|b|^{-3m\beta}\phi\left(\frac{b^m x, 0, v_2, \dots, v_n}{|b|^{3\beta}}\right) : 0 \leq m \\ < i\}\end{aligned} \tag{27}$$

Proof. Let $y = 0$ in inequality (26) and also divide it by $|b|^{3\beta}$, we obtain:

$$\left\|\frac{f(bx)}{b^3} - f(x), v_2, \dots, v_n\right\|_\beta \le \phi\left(\frac{x, 0, v_2, \dots, v_n}{|b|^{3\beta}}\right) \tag{28}$$

now replace x by $b^i x$ in inequality (28) and divide either side by $|b|^{3i}$, then

$$\left\|\frac{f(b^{i+1}x)}{b^{3(i+1)}} - \frac{f(b^i x)}{b^{3i}}, v_2, \dots, v_n\right\|_\beta \le |b|^{-3i\beta}\phi\left(\frac{b^i x, 0, v_2, \dots, v_n}{|b|^{3\beta}}\right) \tag{29}$$

by letting i approach infinity and applying inequality (25), we obtain:

$$\lim_{i\to\infty}\left\|\frac{f(b^{i+1}x)}{b^{3(i+1)}} - \frac{f(b^i x)}{b^{3i}}, v_2, \dots, v_n\right\|_\beta = 0$$

where $x, v_2, \dots, v_n \in X$. Therefore $\{\frac{f(b^i x)}{b^{3i}}\}$ is said to be Cauchy. Now we define another mapping $T: X \to Y$ such that

$$T(x) = \lim_{i\to\infty}\frac{f(b^i x)}{b^{3i}} \tag{30}$$

using the method of mathematical induction one can illustrate that

$$\begin{aligned} &\left\|\frac{f(b^i x)}{b^{3i}} - f(x), v_2, \dots, v_n\right\|_\beta \\ &= \left\|\sum_{m=0}^{i-1}\left(\frac{f(b^{m+1}x)}{b^{3(m+1)}} - \frac{f(b^m x)}{b^{3m}}, v_2, \dots, v_n\right)\right\|_\beta \\ &\le max\left\{\left\|\frac{f(b^{m+1}x)}{b^{3(m+1)}} - \frac{f(b^m x)}{b^{3m}}, v_2, \dots, v_n\right\|_\beta : 0 \le m < i\right\} \\ &\le |b|^{-3m\beta}\, max\left\{\phi\left(\frac{b^m x, 0, v_2, \dots, v_n}{|b|^{3\beta}}\right) : 0 \le m < i\right\} \end{aligned} \tag{31}$$

letting $i \to \infty$ in inequality (31) and applying inequality (30) we can see that the inequality (27) holds. By inequality (26), we have that:

$$\begin{aligned} &\left\|\frac{f\left(b^i(bx+y)\right)}{b^{3i}} + \frac{f\left(b^i(x+by)\right)}{b^{3i}} - (b+1)(b-1)^2\frac{f\left(b^i(x)\right)}{b^{3i}} - (b+1)(b-1)^2\frac{f\left(b^i(y)\right)}{b^{3i}} - (b^2+b)\frac{f\left(b^i(x+y)\right)}{b^{3i}}, v_2, \dots, v_n\right\|_\beta \\ &\le \lim_{i\to\infty}\phi\left(\frac{b^i x, b^i y, v_2, \dots, v_n}{|b|^{3i\beta}}\right) = 0. \end{aligned}$$

Hence the mapping T satisfies Eq. (1). Let $T': X \to Y$ be another function meets inequality (27) then:

$$\|T(x) - T'(x), v_2, \dots, v_n\|_\beta = \|T(b^k x) - T'(b^k x), v_2, \dots, v_n\|_\beta \le max\left\{\|T(b^k x) - f(b^k x), v_2, \dots, v_n\|_\beta, \|f(b^k x) - T'(b^k x), v_2, \dots, v_n\|_\beta\right\} \le \frac{1}{|b|^{3\beta}}\lim_{i\to\infty} max\left\{\phi\left(\frac{b^m x, 0, v_2, \dots, v_n}{|b|^{3mk\beta}}\right) : k \le m < k+i\right\}.$$

which tends to zero as $k \to \infty$.

Theorem 3.7 Let $\phi: X^{n+1} \to \mathbb{R}^+$ be a function with:

$$\lim_{i\to\infty}|b|^{3i\beta}\phi\left(\frac{x}{b^i}, \frac{y}{b^i}, v_2, \dots v_n\right) = 0 \tag{32}$$

for $x, y, v_2, \dots, v_n \in X$. Presume that $f: X \to Y$ (where X is a n.A (n, β)- normed space and Y is a n.A (n, β)- Banach space) is a mapping that meets the inequality:

$$\|D_f(x, y), v_2, \dots, v_n\|_\beta \le \phi(x, y, v_2, \dots, v_n) \tag{33}$$

where, $D_f(x, y)$ is defined as: $f(bx + y) + f(x + by) - (b+1)(b-1)^2[f(x) + f(y)] - b(b+1)f(x+y)$ and where $x, y, v_2, \dots, v_n \in X$. Then another EL cubic type mapping exists uniquely as $T: X \to Y$ such that

$$\begin{aligned} &\|f(x) - T(x), v_2, \dots, v_n\|_\beta \\ &\le \lim_{i\to\infty} max\left\{|b|^{3m\beta}\phi\left(\frac{x}{b^m}, 0, v_2, \dots, v_n\right) : 0 \le m < i\right\} \end{aligned} \tag{34}$$

Proof. Let $y = 0$ in inequality (33), we obtain that:

$$\|f(bx) - b^3 f(x), v_2, \dots v_n\|_\beta \le \phi(x, 0, v_2, \dots, v_n)$$

replace x by $\frac{x}{b^{i+1}}$ and multiply both sides by $|b|^{3i\beta}$, we have:

$$\begin{aligned} &\left\|b^{3i}f\left(\frac{x}{b^i}\right) - b^{3(i+1)}f\left(\frac{x}{b^{i+1}}\right), v_2, \dots, v_n\right\|_\beta \\ &\le |b|^{3i\beta}\phi\left(\frac{x}{b^i}, 0, v_2, \dots, v_n\right) \end{aligned} \tag{35}$$

where $x, v_2, \dots, v_n \in X$. When i approaches ∞ using Eq. (32), we obtain:

$$\lim_{i\to\infty}\left\|b^{3i}f\left(\frac{x}{b^i}\right) - b^{3(i+1)}f\left(\frac{x}{b^{i+1}}\right), v_2, \dots, v_n\right\|_\beta = 0.$$

Therefore, the sequence $\left\{b^{3i}f\left(\frac{x}{b^i}\right)\right\}$ is Cauchy. Now we define the mapping $T: X \to Y$ such that:

$$T(x) = \lim_{i\to\infty} b^{3i}f\left(\frac{x}{b^i}\right) \tag{36}$$

by using the method of mathematical induction, we express that:

$$\begin{aligned} &\left\|f(x) - b^{3i}f\left(\frac{x}{b^i}\right), v_2, \dots, v_n\right\|_\beta \\ &= \left\|\sum_{m=0}^{i-1}\left(b^{3m}f\left(\frac{x}{b^m}\right) - b^{3(m+1)}f\left(\frac{x}{b^{m+1}}\right), v_2, \dots, v_n\right)\right\|_\beta \\ &\le \max\left\{\left\|b^{3m}f\left(\frac{x}{b^m}\right) - b^{3(m+1)}f\left(\frac{x}{b^{m+1}}\right), v_2, \dots, v_n\right\|_\beta : 0 \le m < i\right\} \\ &\le \max\left\{|b|^{3m\beta}\phi\left(\frac{x}{b^m}, 0, v_2, \dots, v_n\right) : 0 \le m < i\right\} \end{aligned} \tag{37}$$

letting $i \to \infty$ in inequality (37) and applying Eq. (36) we can see that the inequality (34) holds. By (33):

$$\begin{aligned} &\left\|b^{3i}f\left(\frac{bx+y}{b^i}\right) + b^{3i}f\left(\frac{x+by}{b^i}\right) - (b+1)(b-1)^2 b^{3i}f\left(\frac{x}{b^i}\right)\right. \\ &\quad - (b+1)(b-1)^2 b^{3i}f\left(\frac{y}{b^i}\right) \\ &\quad \left.- (b^2+b)b^{3i}f\left(\frac{x+y}{b^i}\right)\right\|_\beta \\ &\le \lim_{i\to\infty}|b|^{3i\beta}\phi\left(\frac{x}{b^i}, \frac{y}{b^i}, v_2, \dots v_n\right) = 0 \end{aligned}$$

hence the mapping T satisfies Eq. (1). Let $T': X \to Y$ be another function meets (34), then:

$$\begin{aligned}\|T(x) - T'(x), v_2, \dots, v_n\|_\beta &= \left\|T\left(\frac{x}{b^k}\right) - T'\left(\frac{x}{b^k}\right), v_2, \dots, v_n\right\|_\beta \\ &\le max\left\{\left\|T\left(\frac{x}{b^k}\right) - f\left(\frac{x}{b^k}\right), v_2, \dots v_n\right\|_\beta, \left\|f\left(\frac{x}{b^k}\right) - T'\left(\frac{x}{b^k}\right), v_2, \dots v_n\right\|_\beta\right\} \\ &\le \lim_{i\to\infty} max\left\{\begin{matrix}|b|^{3mk\beta}\phi\left(\frac{x}{b^m}, 0, v_2, \dots, v_n\right) \\ : k \le m < k+i\end{matrix}\right\}.\end{aligned}$$

which tends to zero as $k \to \infty$.

In the following, we give a counter example to the stability result of the above Eq. (1) for the n.A case. For, we consider $b = 3$ in Eq. (1).

Example 3.8 Let $f: X \to \mathbb{R}$ where X is a linear space over K with a non-trivial n.A valuation $(|.|_p)$. Then for $p > 13$ and $f(x) = 2$, $\delta = 1$ we have,

$$|f(3x + y) + f(x + 3y) - 12f(x + y) - 16f(x) - 16f(y)|_{\mathrm{p}} \le \delta$$

by using the fact that $|84|_{\mathrm{p}} = 1$, we have the sequence:

$$\begin{aligned}&\left|27^{-n}f(3^n x) - 27^{-(n+1)}f(3^{n+1}x)\right|_{\mathrm{p}} \\ &= \left|27^{-n}.2 - 27^{-(n+1)}.2\right|_{\mathrm{p}} = \left|27^{-n}\frac{52}{27}\right|_p \\ &= 1\end{aligned}$$

now consider another sequence:

$$\begin{aligned}&\left|27^n f(3^{-n}x) - 27^{(n+1)}f\left(3^{-(n+1)}x\right)\right|_p \\ &= |27^n.2 - 27^{n+1}.2|_{\mathrm{p}} = |27^n(-52)|_p \\ &= 1\end{aligned}$$

thus, the sequences are not convergent.

4. CONCLUSIONS

In this research article, we have examined the GHUR stability of an EL cubic type Eq. (1) in n.A quasi-Banach spaces, n.A p-Banach spaces and n.A (n,β) Banach spaces. Also, we explored the results using famous Hyers direct method. This method is pivotal for achieving the stability outcomes. Additionally, we provided an illustrative example to help the readers demonstrate the application of our results in non-Archimedean context. In future, in the setting of other normed spaces, the GHUR stability for the Eq. (1) or any generalized form of functional equation can be investigated.

REFERENCES

[1] Ulam, S.M. (1940). Problems in Modern Mathematics. Science Editions. John Wiley and Sons, New York, USA, Chapter 6.

[2] Hyers, D.H. (1941). On the stability of the linear functional equation. Proceedings of the National Academy of Sciences, 27(4): 222-224. https://doi.org/10.1073/pnas.27.4.222

[3] Aoki, T. (1950). On the stability of the linear transformation in Banach spaces. Journal of The Mathematical Society of Japan, 2(1-2): 64-66. https://doi.org/10.2969/jmsj/00210064

[4] Rassias, T.M. (1978). On the stability of the linear mapping in Banach spaces. Proceedings of the American Mathematical Society, 72(2): 297-300. https://doi.org/10.2307/2042795

[5] Gajda, Z. (1991). On stability of additive mappings. International Journal of Mathematics and Mathematical Sciences, 14: 431-434. https://doi.org/10.1155/S016117129100056X

[6] Gavruta, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. Journal of Mathematical Analysis and Applications, 184(3): 431-436. http://doi.org/10.1006/jmaa.1994.1211

[7] Kenary, H.A. (2009). On the stability of a cubic functional equation in random normed spaces. Journal of Mathematical Extension, 4(1): 105-113.

[8] Jun, K.W., Kim, H.M. (2002). The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. Journal of Mathematical Analysis and Applications, 274(2): 867-878. https://doi.org/10.1016/S0022-247X(02)00415-8

[9] Jun, K.W., Kim, H.M. (2006) Stability problem for Jensen–type functional equations of cubic mappings. Acta Mathematica Sinica, 22: 1781-1788. https://doi.org/10.1007/s10114-005-0736-9

[10] Ramachandran, A., Sangeetha, S. (2024). Generalized quadratic functional equation and its stability over Non-Archimedean Normed Space. Mathematical Modelling of Engineering Problems, 11(2): 486-492. https://doi.org/10.18280/mmep.110220

[11] Moradlou, F., Najati, A., Vaezi, H. (2009). Stability of homomorphisms and derivations on C^*-Ternary Rings associated to an Euler-Lagrange type additive mapping. Results in Mathematics, 55: 469. https://doi.org/10.1007/s00025-009-0410-0

[12] Pietrzyk, A. (2006). Stability of the Euler-Lagrange-Rassias functional equation. Demonstratio Mathematica, 39(3): 523-530. https://doi.org/10.1515/dema-2006-0307

[13] Najati, A., Moradlou, F. (2009). Stability of an Euler-Lagrange type cubic functional equation. Turkish Journal of Mathematics, 33(1): 65-73. https://doi.org/10.3906/mat-0802-20

[14] Eskandani, G.Z., Gavruta, P., Rassias, J.M., Ramazan, Z. (2011). Generalized Hyers-Ulam stability for a general mixed functional equation in Quasi-β-normed spaces. Mediterranean Journal of Mathematics, 8: 331-348. https://doi.org/10.1007/s00009-010-0082-8

[15] Gordji, M.E., Azadi Kenary, H., Rezaei, H., Lee, Y.W., Kim, G.H. (2012). Solution and Hyers-Ulam-Rassias stability of generalized mixed type additive-quadratic functional equations in fuzzy Banach spaces. Abstract and Applied Analysis, 2012: 953938. https://doi.org/10.1155/2012/953938

[16] Koh, H., Kang, D. (2013). Solution and stability of Euler-Lagrange-Rassias quartic functional equations in various quasi normed spaces. Abstract and Applied Analysis. https://doi.org/10.1155/2013/908168

[17] Hensel, K. (1897). Über eine neue Begründung der

theorie der algebraischen Zahlen. Jahresbericht der Deutschen Mathematiker-Vereinigung, 6: 83-88. http://eudml.org/doc/144593.
[18] Arriola, L.M., Beyer, W.A. (2005). Stability of the Cauchy functional equation over p-adic fields. Real Analysis Exchange, 31(1): 125-132. http://doi.org/10.14321/realanalexch.31.1.0125
[19] Kaiser, Z. (2006). On stability of the monomial functional equation in normed spaces over fields with valuation. Journal of Mathematical Analysis and Applications, 322(2): 1188-1198. https://doi.org/10.1016/j.jmaa.2005.04.087
[20] Van An, L., Tamilvanan, K., Udhayakumar, R., Kabeto, M.J., Van Ngoc, L. (2022). Ulam-Hyers stability results of-quadratic functional equation with three variables in non-Archimedean Banach space and non-Archimedean random normed space. Journal of Function Spaces, 2022. https://doi.org/10.1155/2022/6795978
[21] Elumalai, P., Sangeetha, S., Selvan, A.P. (2024). Fixed point approach to the stability of a cubic and quartic mixed type functional equation in non-Archimedean spaces. Journal of Mathematics and Computer Science, 33(2): 124-136. http://doi.org/10.22436/jmcs.033.02.01
[22] Moslehian, M.S., Rassias, T.M. (2007). Stability of functional equations in non-Archimedean spaces. Applicable Analysis and Discrete Mathematics, 1(2): 325-334. http://doi.org/10.2298/AADM0702325M
[23] Mirmostafaee, A.K. (2009). Stability of quartic mappings in non-Archimedean normed spaces. Kyungpook Mathematical Journal, 49(2): 289-297. http://doi.org/10.5666/KMJ.2009.49.2.289
[24] Mirmostafaee, A.K. (2009). Approximately additive mappings in non-Archimedean normed spaces. Bulletin of the Korean Mathematical Society, 46: 387-400. https://doi.org/10.4134/BKMS.2009.46.2.387
[25] Elumalai, P., Sangeetha, S. (2024). Stability of cubic n-dimensional functional equation in non-Archimedean Banach Spaces. IAENG International Journal of Applied Mathematics, 54(1): 92-99.
[26] Balaanandhan, R., Uma, J. (2023). Fixed point results in partially ordered ultrametric space via p-adic distance. IAENG. International Journal of Applied Mathematics, 53: 772-778.
[27] Eskandani, G.Z., Gavruta, P. (2011). Stability of the Pexiderized Cauchy functional equation in non-Archimedean spaces. Functional Equations in Mathematical Analysis, 52: 307-318. https://doi.org/10.1007/978-1-4614-0055-4_24
[28] Eshaghi Gordji, M., Ghaemi, M.B., Alizadeh, B. (2011). A fixed-point method for perturbation of higher ring derivations in non-Archimedean Banach algebras. International Journal of Geometric Methods in Modern Physics, 8(7): 1611-1625. https://doi.org/10.1142/S021988781100583X
[29] Park, W.G. (2011). Approximate additive mappings in 2-Banach spaces and related topics. Journal of Mathematical Analysis and Applications, 376(1): 193-202. https://doi.org/10.1016/j.jmaa.2010.10.004
[30] Ciepliński, K. (2021). Ulam stability of functional equations in 2-Banach spaces via the fixed-point method. Journal of Fixed-Point Theory and Applications, 23(3): 33. https://doi.org/10.1007/s11784-021-00869-x
[31] Wang, Z.H., Park, C., Shin, D.Y., (2021). Additive ρ-functional inequalities in non-Archimedean 2-normed spaces. AIMS Mathematics, 6(2): 1905-1919. https://doi.org/10.3934/math.2021116
[32] Aribou, Y., Kabbaj, S. (2021). The stability of N-dimensional quadratic functional inequality in non-Archimedean Banach spaces. São Paulo Journal of Mathematical Sciences, 16: 1382-1400. https://doi.org/10.1007/s40863-021-00220-9
[33] Wang, Z. (2021). Approximate mixed type quadratic-cubic functional equation. AIMS Mathematics, 6(4): 3546-3561. https://doi.org/10.3934/math.2021211
[34] Alessa, N., Tamilvanan, K., Loganathan, K., Selvi, K.K. (2021). Hyers-Ulam stability of functional equation deriving from quadratic mapping in non-Archimedean (n,β) - Normed spaces. Journal of Function Spaces, 2021: 1-10. https://doi.org/10.1155/2021/9953214
[35] Yang, X.Z., Chang, L.D., Liu, G.F., Shen, G.N. (2015). Stability of functional equations in (n, β) -normed spaces. Journal of Inequalities and Applications, 112. https://doi.org/10.1186/s13660-015-0628-1
[36] Ramakrishnan, K., Uma, J. (2023). Stability of a quadratic-additive functional equation in non-Archimedean quasi Banach spaces. In AIP Conference Proceedings. AIP Publishing, 2852(1). https://doi.org/10.1063/5.0164556
[37] Ramakrishnan, K., Uma, J., Ponmana Selvan, A. (2023). Generalized Hyers-Ulam stability of a bi-quadratic mapping in non-Archimedean spaces. Journal of Mathematics and Computer Science, 31(4): 393-402. http://doi.org/10.22436/jmcs.031.04.04
[38] Jun, K.W., Kim, H.M. (2007). On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces. Journal of Mathematical Analysis and Applications, 332(2): 1335-1350. https://doi.org/10.1016/j.jmaa.2006.11.024
[39] Dung, N.V., Sintunavarat, W. (2022). Improvements on the stability of Euler-Lagrange type cubic maps in quasi-Banach spaces. Analysis Mathematica, 48: 69-84. https://doi.org/10.1007/s10476-021-0109-3

NOMENCLATURE

κ	modulus of concavity

Greek symbols

ϕ, τ	functions
δ	smallest positive integer