



## Approximate Solution of Linear Fredholm Integral Equation of the Second Kind Using Modified Simpson's Rule

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### ABSTRACT

In this paper, we introduce a novel approach to obtain approximate numerical solutions of linear Fredholm integral equations of the second kind. This method is founded on the modified Simpson's quadrature rule. We transform the Fredholm integral equation of the second kind into a system of linear equations and provide numerical examples. Numerical results were compared and interpreted with tables and graphs and the solution was shown to be consistent. Furthermore, we conduct a comparative analysis, comparing the absolute error in the solution with existed methods This comparison serves to highlight the efficiency and accuracy of our proposed method.

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*linear Fredholm integral equation, modified Simpson's quadrature rule, error approximation*

### 1. INTRODUCTION

Fredholm integral equations are a class of mathematical equations that arise in various branches of applied mathematics and engineering. These equations play a significant role in the study of boundary value problems, signal processing, quantum mechanics, and many other areas [1-3]. So approximate solution of linear Fredholm integral equation of the second kind is one of the most important subjects discussed in recent years, using different kind of methods such as Method of Moments, Variational Iteration Method [4].

Various methods, including iterative procedures, eigenfunction expansions, and numerical quadrature, are employed to find solutions to these equations. Many authors have extensively explored the approximate solutions of linear Fredholm integral equations of the second kind, utilizing a variety of quadrature methods. Such that Haar wavelets [5], hybrid functions [6], sinc-Galerkin method [7], sinc-collocation method [8], collocation methods based on cubic spline quadrature [9], collocation methods based on B-cubic spline quadrature [10], repeated modified trapezoid quadrature method [11, 12], Numerical solution of the Fredholm integral equations with a quadrature method [13], Bernoulli wavelet based numerical method for solving Fredholm integral equations of the second kind [14]. Malmir [15] used the numerical solution method based on Chebyshev and Legendre polynomials to solve the Fredholm integral equation of the Second kind.

Each of these approaches offers unique advantages and has contributed to our understanding and practical implementation of approximate solutions for such integral equations.

In this study, we introduce a novel approach to obtain an approximate numerical solution of linear Fredholm integral equation of the second kind, which is given by the equation:

$$\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x), a \leq x \leq b \quad (1)$$

where, the function  $f(x)$  and the kernel  $k(x, t)$  are known, but  $\phi(x)$  the exact solution is unknown and to be determined. We assume that the integral equation (1) has a unique solution. To find this solution numerically, we employ a modified numerical technique based on the quadrature Simpson's rule, this method is first proposed by Nadir and Rahmoune [16], for solving Volterra integral equations. The idea is to approximate the solution of the linear Fredholm integral equations in even number of equally spaced points. This technique will enable us to approximate the solution  $\phi(x)$  efficiently and accurately.

The paper is organized into six sections. The second section presents the existence and uniqueness of a solution. In the third section, we delve into the algorithm used in our research method, providing a detailed explanation of its workings. A study of convergence is conducted in the fourth section. Section 5 is devoted to presenting a series of diverse examples that serve as practical tests for evaluating the method's performance and applicability. Finally, in the last section, conclusion of the proposed method is discussed.

### 2. EXISTENCE AND UNIQUENESS OF SOLUTION

In this section, we give some conditions which ensure the existence and uniqueness of solutions of linear Fredholm integral equations.

#### Definition

Let  $f: [a, b] \rightarrow \mathbb{R}$ , and  $k: [a, b] \times [a, b] \rightarrow \mathbb{R}$  be both continuous functions, and  $\phi_0: [a, b] \rightarrow \mathbb{R}$ , is an arbitrary continuous function, we define the sequence of functions by:  $\phi_n(x) = \int_a^b k(x, t)\phi_{n-1}(t)dt + f(x)$ ,  $n \geq 1, x \in [a, b]$ .

This sequence called Fredholm sequence.

**Lemma**

If  $\phi_0: [a, b] \rightarrow \mathbb{R}$  is an arbitrary continuous function and  $\phi_n(x) = \int_a^b k(x, t)\phi_{n-1}(t)dt$ , for all  $n \geq 1, x \in [a, b]$ .

Then:

$$1. |\phi_n(x)| \leq M^n \max_{a \leq x \leq b} |\phi_0(x)| \text{ for all } n \geq 1, x \in [a, b],$$

where,  $M = \max_{a \leq x \leq b} \int_a^b |k(x, t)|dt$ .

2. If  $M < 1$ , then the sequence  $(\phi_n)$  uniformly converges to the zero function for all  $x \in [a, b]$ .

**Proof:** Application the inductive proof [17].

**Theorem 1**

Let  $f: [a, b] \rightarrow \mathbb{R}$ , and  $k: [a, b] \times [a, b] \rightarrow \mathbb{R}$  be both continuous functions, and  $M = \max_{a \leq x \leq b} \int_a^b |k(x, t)|dt$ . If  $M < 1$ , then:

1.  $\phi=0$  is the unique solution to homogeneous equation  $\phi(x) = \int_a^b k(x, t)\phi(t)dt$ .

2. The equation  $\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x)$  has at most one solution.

3. Let  $\phi_0=f$ , and define the sequence of functions by  $\phi_n(x) = \int_a^b k(x, t)\phi_{n-1}(t)dt + f(x)$ ,  $n \geq 1$ .

The sequence  $(\phi_n)$  converges to the unique solution of the equation  $\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x)$ .

**Proof:** The proof for this theorem can be found in reference [18].

**Theorem 2 (Fredholm Alternative Theorem)**

Consider the linear Fredholm integral equation of the second kind  $\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x)$ .

If the homogeneous Fredholm integral equation of the second kind  $\phi(x) = \int_a^b k(x, t)\phi(t)dt$  has only the trivial solution  $\phi(x)=0$ , then the corresponding non-homogeneous Fredholm integral equation of the second kind  $\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x)$  has a unique solution for any right-hand side  $f$ .

**3. DEVELOPMENT OF THE METHOD**

Consider the linear Fredholm integral equation of the second kind with regular kernels (non- singular):

$$\phi(x) - \int_a^b k(x, t)\phi(t)dt = f(x), a \leq x \leq b \tag{2}$$

where,  $a, b \in \mathbb{R}$ ,  $f(x)$  is a known continuous function on the finite interval  $[a, b]$ , the kernel  $k(x, t)$  is known and continuous function in  $[a, b] \times [a, b]$  but  $\phi(x)$  the exact solution of the equation (1) is an unknown continuous function in  $[a, b]$  to be determined.

Let  $a = x_0 < x_1 < \dots < x_{2i-1} < x_{2i} < \dots < x_{2n} = b$ , be a subdivision of the interval  $[a, b]$ , with  $h = \frac{b-a}{2n}$ .

We approximate the right-hand integral (1) at the even point  $(x_{2i})$  with the Simpson's quadrature rule, we have:

$$\begin{aligned} \phi(x_{2i}) - \int_a^b k(x_{2i}, t)\phi(t)dt &= f(x_{2i}) \\ \phi(x_{2i}) - \sum_{j=0}^{j=n-1} \int_{x_{2j}}^{x_{2j+2}} k(x_{2i}, t)\phi(t)dt &= f(x_{2i}), \end{aligned}$$

$$\begin{aligned} i=0, 1, \dots, n \\ \phi_{2i} - \frac{h}{3} \sum_{j=0}^{j=n-1} (k_{2i,2j}\phi_{2j} + 4k_{2i,2j+1}\phi_{2j+1} + \\ k_{2i,2j+2}\phi_{2j+2}) = f_{2i} \end{aligned}$$

For a smaller step  $h$ , an approximation to  $\phi_{2i}$  can be computed by replacing  $\phi_{2j+1}$  by the average  $\frac{\phi_{2j} + \phi_{2j+2}}{2}$ .

$$\begin{aligned} \phi_{2i} - \frac{h}{3} \sum_{j=0}^{j=n-1} (k_{2i,2j}\phi_{2j} + 4k_{2i,2j+1} \frac{\phi_{2j} + \phi_{2j+2}}{2} + \\ k_{2i,2j+2}\phi_{2j+2}) = f_{2i} \end{aligned}$$

or

$$\begin{aligned} \phi_{2i} - \frac{h}{3} \sum_{j=0}^{j=n-1} [(k_{2i,2j} + 2k_{2i,2j+1})\phi_{2j} + (k_{2i,2j+2} + \\ 2k_{2i,2j+1})\phi_{2j+2}] = f_{2i} \end{aligned}$$

then

$$\begin{aligned} \phi_{2i} - \frac{h}{3} \sum_{j=0}^{j=n-1} (k_{2i,2j} + 2k_{2i,2j+1})\phi_{2j} - \frac{h}{3} \sum_{j=0}^{j=n-1} (k_{2i,2j+2} + \\ 2k_{2i,2j+1})\phi_{2j+2} = f_{2i} \\ \phi_{2i} - \frac{h}{3} \sum_{j=0}^{j=n-1} (k_{2i,2j} + 2k_{2i,2j+1})\phi_{2j} - \frac{h}{3} \sum_{j=1}^{j=n} (k_{2i,2j} + \\ 2k_{2i,2j-1})\phi_{2j} = f_{2i} \end{aligned}$$

for  $i=0, \dots, n$ , we get the following system:

$$\begin{aligned} \phi_{2i} - \frac{h}{3} (k_{2i,0} + 2k_{2i,1})\phi_0 - \frac{2h}{3} \sum_{j=1}^{j=n-1} (k_{2i,2j-1} + k_{2i,2j} + \\ k_{2i,2j+1})\phi_{2j} - \frac{h}{3} (2k_{2i,2n-1} + k_{2i,2n})\phi_{2n} = f_{2i} \end{aligned}$$

With  $2n+1$  equation, and  $2n+1$  unknowns.

**4. ERROR ESTIMATION**

In the subinterval  $[x, x+2h]$ . we have:

$$\begin{aligned} \int_s^{s+2h} k(x, t)\phi(t)dt = \frac{h}{3} (k(x, s)\phi(s) + 4k(x, s+h)\phi(s+h) \\ + k(x, s+2h)\phi(s+2h)) + \frac{h^5}{90} (k(x, \xi)\phi(\xi)), \text{ so} \end{aligned}$$

$$\begin{aligned} \int_a^b k(x, t)\phi(t)dt = \sum_{i=0}^n \int_{x_{2i}}^{x_{2(i+1)}} k(x, t)\phi(t)dt = \\ \frac{h}{3} \sum_{i=0}^n (k(x, x_{2i})\phi(x_{2i}) + 4k(x, x_{2i+1})\phi(x_{2i+1}) + \\ k(x, x_{2i})\phi(x_{2i})) + n \left( \frac{h^5}{90} (k(x, \xi)\phi(\xi)) \right) \\ = \frac{h}{3} \sum_{i=0}^n (k(x, x_{2i})\phi(x_{2i}) + 4k(x, x_{2i+1})\phi(x_{2i+1}) + \\ k(x, x_{2i})\phi(x_{2i})) + (b-a) \frac{h^4}{180} (k(x, \xi)\phi(\xi)) \end{aligned}$$

Also, noting that, if the segment width  $h$  is halved to  $h/2$ , then:

$$\begin{aligned} E(h/2) = 2((b-a) \frac{(h/2)^4}{180} k(x, \xi)\phi(\xi)) = \frac{1}{16} ((b-a) \\ a) \frac{(h)^4}{90} k(x, \xi)\phi(\xi) = \frac{1}{8} E(h) \end{aligned}$$

**5. ILLUSTRATING EXAMPLES**

The objective of this section is to demonstrate the efficacy of the method outlined in the paper through a series of

numerical experiments.

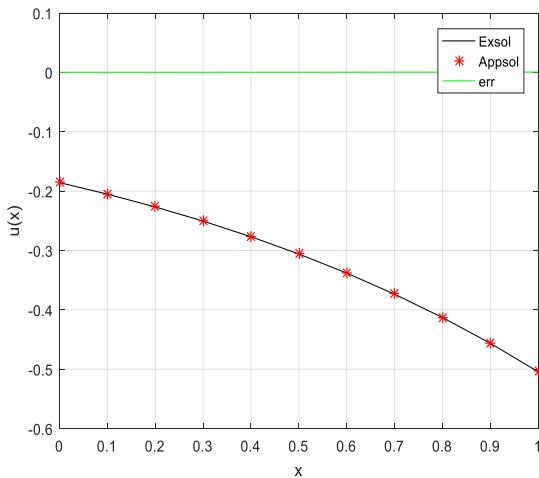
**Example 1.**

Let us consider the Fredholm integral equation of the second kind  $\phi(x) - \int_0^1 2 \exp(x+t)\phi(t)dt = \exp(x)$ ,  $0 \leq x \leq 1$ , the exact solution  $\phi_{ex}(x)$  is given by  $\phi_{ex}(x) = \frac{\exp(x)}{2-\exp(2)}$ .

Table 1 and Figure 1 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=10$  of the equation in the Example 1 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 1.** Numerical results for  $n=10$  in example 1

Val. of x	Ex.sol	App.sol	Error
0.00	-0.185561	-0.185378	1.830029e-04
0.10	-0.205077	-0.204875	2.022495e-04
0.20	-0.226645	-0.226422	2.235202e-04
0.30	-0.250481	-0.250234	2.470281e-04
0.40	-0.276825	-0.276552	2.730082e-04
0.50	-0.305939	-0.305637	3.017207e-04
0.60	-0.338115	-0.337781	3.334530e-04
0.70	-0.373674	-0.373306	3.685226e-04
0.80	-0.412974	-0.412567	4.072804e-04
0.90	-0.456407	-0.455957	4.501145e-04
1.00	-0.504408	-0.503910	4.974534e-04

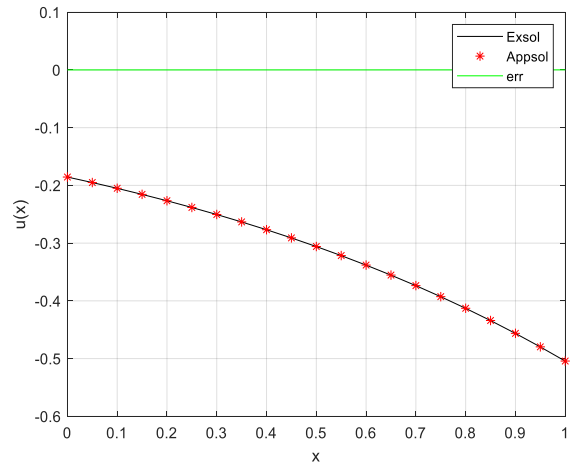


**Figure 1.** Exact, approximate solutions and absolute error ( $n=10$ ) for example 1

Table 2 and Figure 2 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=20$  of the equation in the Example 1 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 2.** Numerical results for  $n=20$  in example 1

Val. of x	Ex.sol	App.sol	Error
0.00	-0.185561	-0.185515	4.581175e-05
0.10	-0.205077	-0.205026	5.062981e-05
0.20	-0.226645	-0.226589	5.595460e-05
0.30	-0.250481	-0.250420	6.183939e-05
0.40	-0.276825	-0.276757	6.834310e-05
0.50	-0.305939	-0.305863	7.553081e-05
0.60	-0.338115	-0.338031	8.347445e-05
0.70	-0.373674	-0.373582	9.225354e-05
0.80	-0.412974	-0.412872	1.019559e-04
0.90	-0.456407	-0.456294	1.126787e-04
1.00	-0.479808	-0.504283	1.245292e-04



**Figure 2.** Exact, approximate solutions and absolute error ( $n=20$ ) for example 1

**Example 2.**

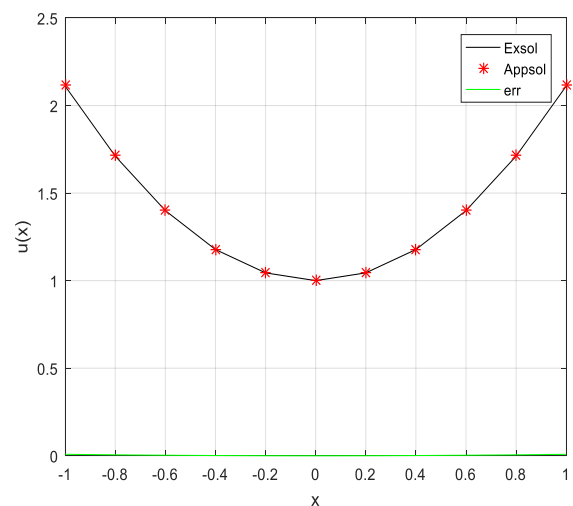
Consider the linear Fredholm integral equation of the second kind  $\phi(x) - \int_{-1}^1 (xt + x^2t^2)\phi(t)dt = 1$ ,  $-1 \leq x \leq 1$ .

1. The exact solution  $\phi_{ex}(x)$  is given by  $\phi_{ex}(x) = 1 + \frac{10}{9}x^2$ .

Table 3 and Figure 3 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=10$  of the equation in the Example 2 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 3.** Numerical results for  $n=10$  in example 2

Values of x	Ex.sol	App.sol	Error
-1.00	2.111111	2.119370	8.258460e-03
-0.80	1.711111	1.716397	5.285414e-03
-0.60	1.400000	1.402973	2.973046e-03
-0.40	1.177778	1.179099	1.321354e-03
-0.20	1.044444	1.044775	3.303384e-04
0.00	1.000000	1.000000	0.000000e+00
0.20	1.044444	1.044775	3.303384e-04
0.40	1.177778	1.179099	1.321354e-03
0.60	1.400000	1.402973	2.973046e-03
0.80	1.711111	1.716397	5.285414e-03
1.00	2.111111	2.119370	8.258460e-03

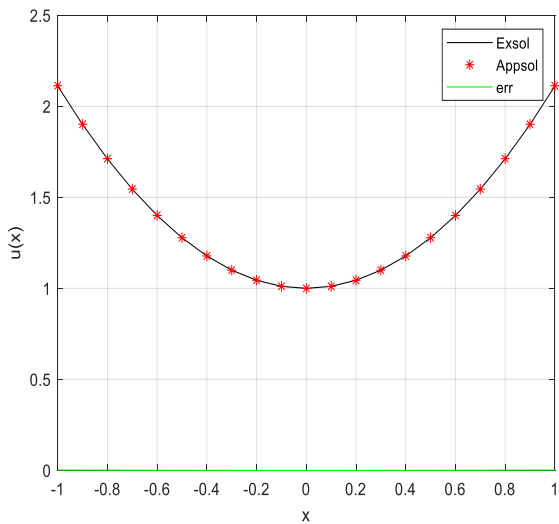


**Figure 3.** Exact, approximate solutions and absolute error ( $n=10$ ) for example 2

Table 4 and Figure 4 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=20$  of the equation in the Example 2 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 4.** Numerical results for  $n=20$  in example 2

Val. of x	Ex.sol	App.sol	Error
-1.00	2.111111	2.113170	2.059365e-03
-0.80	1.711111	1.712429	1.317994e-03
-0.60	1.400000	1.400741	7.413715e-04
-0.40	1.177778	1.178107	3.294985e-04
-0.20	1.044444	1.044527	8.237462e-05
0.00	1.000000	1.000000	0.000000e+00
0.20	1.044444	1.044527	8.237462e-05
0.40	1.177778	1.178107	3.294985e-04
0.60	1.400000	1.400741	7.413715e-04
0.80	1.711111	1.712429	1.317994e-03
1.00	2.111111	2.113170	2.059365e-03



**Figure 4.** Exact, approximate solutions and absolute error ( $n=20$ ) for example 2

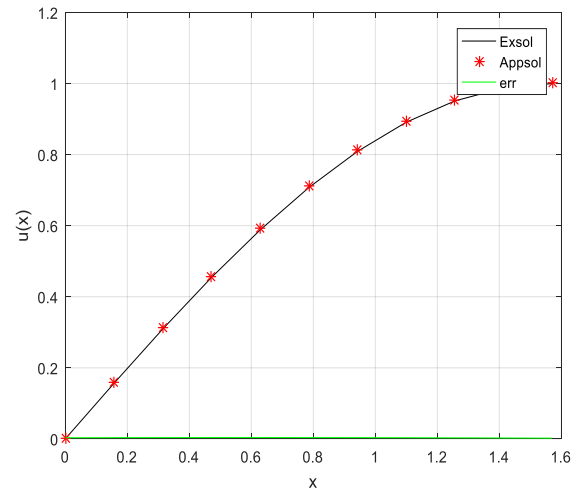
**Example 3.**

Consider the linear Fredholm integral equation of the second kind  $\phi(x) - \int_0^{\frac{\pi}{2}} \frac{4}{\pi} \cos(x-t)\phi(t)dt = -\frac{2}{\pi} \cos(x)$ ,  $0 \leq x \leq \frac{\pi}{2}$ , the exact solution  $\phi_{ex}(x)$  is given by  $\phi_{ex}(x)=\sin(x)$ .

Table 5 and Figure 5 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=10$  of the equation in the Example 3 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 5.** Numerical results for  $n=10$  in example 3

Val. of x	Ex.sol	App.sol	Error
0.00	0.000000	0.003242	3.241537e-03
0.157	0.156434	0.159961	3.526219e-03
0.314	0.309017	0.312741	3.724075e-03
0.471	0.453990	0.457821	3.830231e-03
0.628	0.587785	0.591627	3.842074e-03
0.785	0.707107	0.710866	3.759312e-03
0.942	0.809017	0.812601	3.583984e-03
1.100	0.891007	0.894327	3.320406e-03
1.257	0.951057	0.954032	2.975069e-03
1.414	0.987688	0.990245	2.556476e-03
1.571	1.000000	1.002075	2.074933e-03

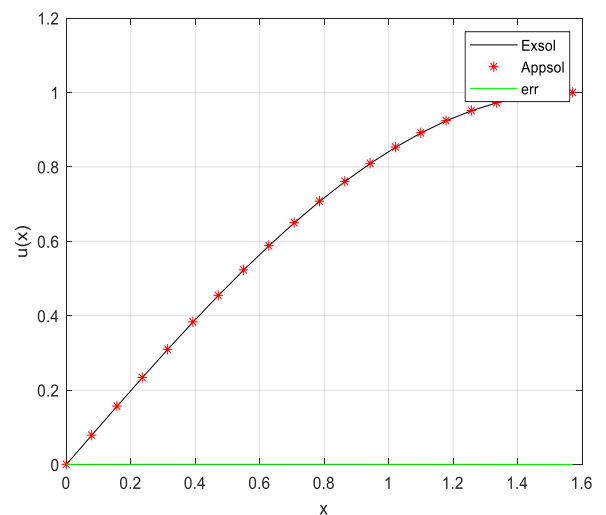


**Figure 5.** Exact, approximate solutions and absolute error ( $n=10$ ) for example 3

Table 6 and Figure 6 present the exact, approximate solutions and the absolute error  $|\phi_{app}-\phi_{ex}(x)|$  for  $n=20$  of the equation in the Example 3 obtained by the modified Simpson's rule, in some arbitrary points.

**Table 6.** Numerical results for  $n=20$  in example 3

Val. of x	Ex.sol	App.sol	Error
0.00	0.000000	0.000808	8.081830e-04
0.157	0.156434	0.157313	8.788296e-04
0.314	0.309017	0.309945	9.278365e-04
0.471	0.453990	0.454944	9.539970e-04
0.628	0.587785	0.588742	9.566669e-04
0.785	0.707107	0.708043	9.357805e-04
0.942	0.809017	0.809909	8.918521e-04
1.100	0.891007	0.891832	8.259633e-04
1.257	0.951057	0.951796	7.397366e-04
1.414	0.987688	0.988324	6.352951e-04
1.571	1.000000	1.000515	5.152105e-04



**Figure 6.** Exact, approximate solutions and absolute error ( $n=20$ ) for example 3

In the following examples, a comparison with the least-squares and polynomial method [19] is done.

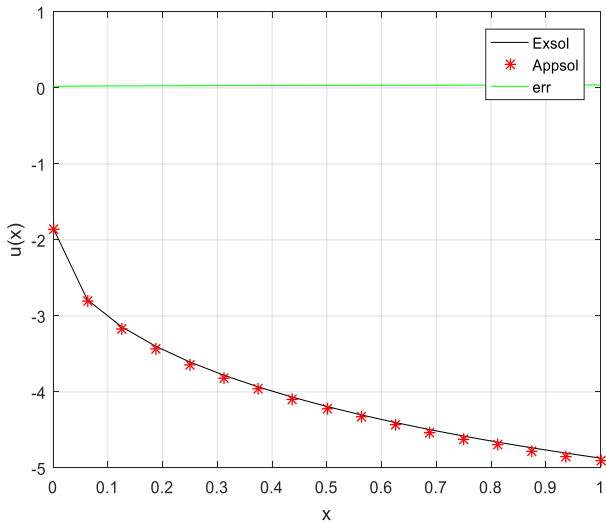
**Example 4.**

Consider the linear Fredholm integral equation of the second kind [19]  $\phi(x) - \int_0^1 (\sqrt{x} + \sqrt{t})\phi(t)dt = 1 + x, 0 \leq x \leq 1$ , the exact solution  $\phi_{ex}(x)$  is given by  $\phi_{ex}(x) = \frac{-129}{70} - \frac{141}{35}\sqrt{x} + x$ .

Numerical results for n=16 of the equation in the Example 4 obtained by the modified Simpson's method, in some arbitrary points are presented in Table 7 and Figure 7, a comparison of obtained results with ones obtained by least-squares [19], is given in Table 7.

**Table 7.** Comparison of results for n=16 in example 4

Values of x	Error	$ \phi - \phi_1 $ [19]
0.000	6.571721e-03	6.848e-01
0.125	8.801403e-03	
0.250	9.724968e-03	9.31e-02
0.375	1.043364e-02	
0.500	1.103109e-02	2.15e-02
0.625	1.155744e-02	
0.750	1.203330e-02	4.26e-02
0.875	1.247091e-02	
1.000	1.287821e-02	1.233e-01

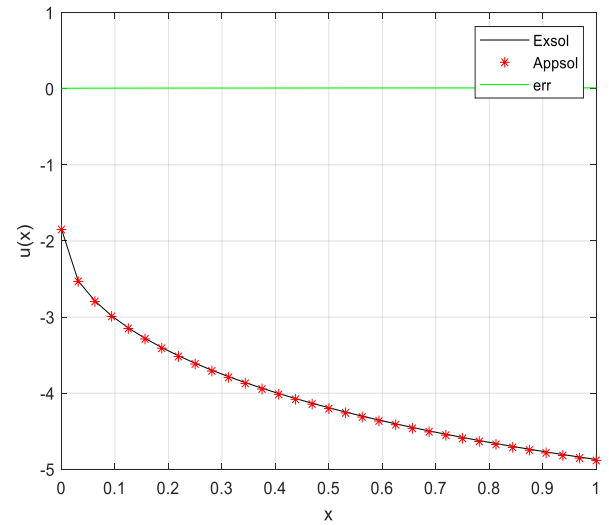


**Figure 7.** Exact, approximate solutions and absolute error (n=16) for example 4

Numerical results for n=32 of the equation in the Example 4 obtained by the modified Simpson's method, in some arbitrary points are presented in Table 8 and Figure 8, a comparison of obtained results with ones obtained by polynomial method  $|\phi - \phi_2|$  [19], is given in Table 8.

**Table 8.** Comparison of results for n=32 in example 4

Values of x	Error	Error [19]
0.000	6.571721e-03	6.927e-01
0.125	8.801403e-03	
0.250	9.724968e-03	8.20e-02
0.375	1.043364e-02	
0.500	1.103109e-02	3.69e-02
0.625	1.155744e-02	
0.750	1.203330e-02	6.38e-02
0.875	1.247091e-02	
1.000	1.287821e-02	9.50e-02



**Figure 8.** Exact, approximate solutions and absolute error (n=32) for example 4

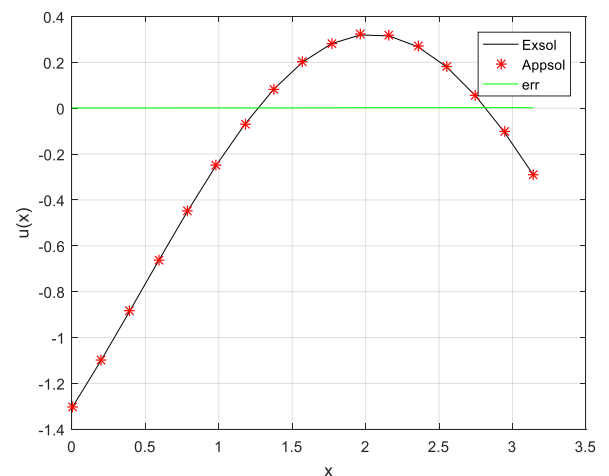
**Example 5.**

In the final example, we apply our approach to the integral equation [19]  $\phi(x) - \int_0^\pi (\cos(x) + \cos(t))\phi(t)dt = \sin(x), 0 \leq x \leq \pi$  to approximate the unique solution  $\phi_{ex}(x) = \sin(x) + \frac{4}{2-\pi^2}\cos(x) + \frac{2\pi}{2-\pi^2}$ .

Numerical results for n=16 of the equation in the Example 5 obtained by the modified Simpson's method, in some arbitrary points are presented in Table 9 and Figure 9, a comparison of obtained results with ones obtained by least-squares [19], is given in Table 9.

**Table 9.** Comparison of results for n=16 in example 5

Values of x	Error	Error [19]
0.00	1.500861e-03	5.228e-01
$\pi/8$	1.532374e-03	
$\pi/4$	1.622115e-03	2.88e-02
$3\pi/8$	1.756422e-03	
$\pi/2$	1.914847e-03	3.619e-01
$5\pi/8$	2.073273e-03	
$3\pi/4$	2.207580e-03	1.166e-01
$7\pi/8$	2.297321e-03	
$\pi$	2.328833e-03	7.548e-01

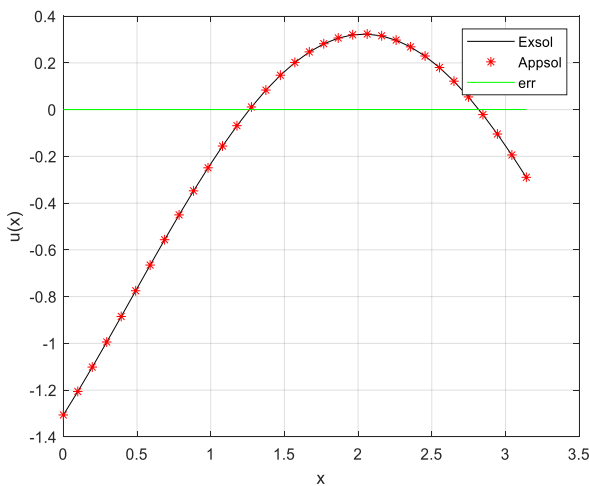


**Figure 9.** Exact, approximate solutions and absolute error (n=16) for example 5

Numerical results for  $n=32$  of the equation in the Example 5 obtained by the modified Simpson's method, in some arbitrary points are presented in Table 10 and Figure 10, a comparison of obtained results with ones obtained by polynomial method  $|\phi-\phi_2|$  [19], is given in Table 10.

**Table 10.** Comparison of results for  $n=32$  in example 5

Values of x	Error	Error [19]
0.00	3.747142e-04	5.125e-01
$\pi/8$	3.826071e-04	
$\pi/4$	4.050841e-04	2.88e-02
$3\pi/8$	4.387234e-04	
$\pi/2$	4.784037e-04	3.664e-01
$5\pi/8$	5.180839e-04	
$3\pi/4$	5.517233e-04	1.182e-01
$7\pi/8$	5.742003e-04	
$\pi$	5.820932e-04	7.548e-01



**Figure 10.** Exact, approximate solutions and absolute error ( $n=32$ ) for example 5

## 6. CONCLUSIONS

A modified method for solving linear Fredholm integral equation of second kind with regular kernel based on the quadrature Simpson's rule was presented. After some experiments, using different forms of kernels, as it was shown in the previous section, when we had mentioned three examples with  $n=10$  then  $n=20$ , and  $n=16$  then  $n=32$  for the two last examples. The comparison of the two last examples with the least-squares and polynomial methods [19] shows the efficiency of this technical for each example we conclude that the method is effective and accurate for solving such kind of equations specially when we increase the value of  $n$ .

Apparently, in order to have the convergence of the present method, both kernel functions  $k(x, t)$  and  $f(x)$  have to be continuous.

## REFERENCES

[1] Wazwaz, A.M. (2011). Linear and nonlinear integral equations: Methods and applications. Springer, New York, NY, USA. <https://doi.org/10.1007/978-3-642-21449-3>

[2] Golberg, M.A. (2013). Numerical solution of integral equations. Springer Science & Business Media. <https://doi.org/10.1007/978-1-4899-2593-0>

[3] Kanwal, R.P. (2013). Linear integral equations. Springer Science & Business Media. <https://doi.org/10.1007/978-1-4612-0765-8>

[4] Ray, S.S., Sahu, P.K. (2013). Numerical methods for solving Fredholm integral equations of second kind. Abstract and Applied Analysis, 2013: 426916. <http://doi.org/10.1155/2013/426916>

[5] Maleknejad, K., Mirzaee, F. (2005). Using rationalized Haar wavelet for solving linear integral equations. Applied Mathematics and Computation, 160(2): 579-587. <https://doi.org/10.1016/j.amc.2003.11.036>

[6] Kajani, M.T., Vencheh, A.H. (2005). Solving second kind integral equations with hybrid Chebyshev and block-pulse functions. Applied Mathematics and Computation, 163(1): 71-77. <https://doi.org/10.1016/j.amc.2003.11.044>

[7] Stenger, F. (1979). A "Sinc-Galerkin" method of solution of boundary value problems. Mathematics of Computation, 33(145): 85-109. <https://doi.org/10.2307/2006029>

[8] Rashidinia, J., Zarebnia, M. (2005). Numerical solution of linear integral equations by using Sinc-collocation method. Applied Mathematics and Computation, 168(2): 806-822. <https://doi.org/10.1016/j.amc.2004.09.044>

[9] Netravali, A.N., de Figueiredo, R.J. (1974). Spline approximation to the solution of the linear Fredholm integral equation of the second kind. SIAM Journal on Numerical Analysis, 11(3): 538-549. <https://doi.org/10.1137/0711045>

[10] Rashidinia, J., Babolian, E., Mahmoodi, Z. (2011). Spline collocation for Fredholm integral equations. Mathematical Sciences, 5(2). <https://www.sid.ir/paper/322537/en>.

[11] Saberi-Nadjafi, J., Heidari, M. (2007). Solving linear integral equations of the second kind with repeated modified trapezoid quadrature method. Applied Mathematics and Computation, 189(1): 980-985. <https://doi.org/10.1016/j.amc.2006.11.165>

[12] Mirzaee, F., Piroozfar, S. (2011). Numerical solution of linear Fredholm integral equations via modified Simpson's quadrature rule. Journal of King Saud University-Science, 23(1): 7-10. <https://doi.org/10.1016/j.jksus.2010.04.011>

[13] Katani, R. (2019). Numerical solution of the Fredholm integral equations with a quadrature method. SeMA Journal, 76(2): 271-276. <https://doi.org/10.1007/s40324-018-0175-z>

[14] Shiralashetti, S.C., Mundewadi, R.A. (2016). Bernoulli wavelet based numerical method for solving Fredholm integral equations of the second kind. Journal of Information and Computing Science, 11(2): 111-119.

[15] Malmir, I. (2016). A comparison of Chebyshev polynomial and Legendre polynomials. International Journal of Scientific Research Publications, 6(3).

[16] Nadir, M., Rahmoune, A. (2007). Modified method for solving linear Volterra integral equations of the second kind using Simpson's rule. International Journal: Mathematical Manuscripts, 1(1): 141-146.

[17] Lin, H. (2023). A primer to linear integral equations of the second kind.

[18] Ledder, G. (1996). A simple introduction to integral

equations. *Mathematics Magazine*, 69(3): 172-181.  
<https://doi.org/10.1080/0025570X.1996.11996422>  
[19] Chakrabarti, A., Martha, S.C. (2009). Approximate

solutions of Fredholm integral equations of the second kind. *Applied Mathematics and Computation*, 211(2): 459-466. <https://doi.org/10.1016/j.amc.2009.01.088>