






## Unreliable Multi Server Retrial Queuing System with Reneging and Diverse Outgoing Services

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### ABSTRACT

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This article explores a M/M/m unreliable retrial queue with reneging and diverse outgoing services. Incoming calls that arrive and discover all servers occupied join the orbit. The buffering incomings from the orbit retry their request after a while or leave the system without receiving service. When the orbit becomes empty, the idle server provides outgoing services. It is assumed that there are two categories of outgoing services. Due to unexpected circumstances, the server may breakdown. When a server undergoes breakdown, immediate repair process begins. Post-server breakdown, incoming calls go into orbit and retry service randomly, whereas the two variants of outgoing calls leave the system. The study utilizes a quasi-birth-death (QBD) process to analyse the stationary system size distribution. The steady state probabilities and the rate matrix are obtained through the matrix geometric method (MGM). Various performance metrics are evaluated for the proposed model. The study examines the impact of various system-based parameters on efficiency metrics with the help of numerical results.

## 1. INTRODUCTION

A queue is formed when all available servers are busy, consisting of customers waiting for service. During the wait, customers can observe the status of the servers. Queues can be found in various settings such as supermarkets, ticket counters, hospitals and etc. Dutta and Choudhury [1] utilized simulation methods to analyse a M/M/1 queues' performance.

A retrial queue is a queuing system where customers who arrive for service but are unable to get prompt service due to occupied servers are directed to an orbit. In the orbit, they cannot observe the status of the servers and must randomly re-arrive for service after some time has elapsed. In a classical retrial queue, the re-arrival times of customers are independent of other customers. For example, in a packet-switched network, packets being sent by the router are temporarily stored in a buffer while they wait for their turn to be transmitted. Kim and Kim [2] have made a comprehensive survey of retrial queueing systems, covering their various aspects and recent developments. Zhang et al. [3] investigated a retrial queue with varying service provision rates based on the server status and proposes an optimal model for the same. Moiseev et al. [4] presented an asymptotic diffusion for a retrial queue with multiple servers, where the service time followed a hyper-exponential distribution, providing a valuable tool for analysing and optimizing such systems. Fiems [5] reviewed retrial queueing models with varying retry durations, using mathematical and probabilistic techniques to understand their

performance and behaviour in practical applications. Arivudainambi and Godhandaraman [6] established the steady-state likelihoods and system effectiveness metrics of a retrial queue, including the anticipated count of customers in the queueing system and the anticipated waiting time.

Reneging is a phenomenon in queuing theory where customers in a queue anticipate a delay in receiving service and subsequently decide to leave the queue without being served. For example, a customer facing an issue on mobile connectivity network may lodge a complaint to the customer care centre. Customers can await service, when all the servers become busy, they may leave the system without getting service. Ding et al. [7] have investigated M/M/1 queues with reneging. Singh et al. [8] have considered finite M/M/1/K with reneging considering fast and slow working phases. Wang and Zhang [9] have investigated the busy period of a queue structure with discouragement of customers.

In the context of service facilities, it is acknowledged that unexpected server interruptions may be encountered due to various reasons like excessive usage, improper handling, and other unforeseen reasons. This unforeseen event may result in the temporary cessation of service delivery. After an effective repair of the inoperable server, the system resumes its normal service operation. Chang and Wang [10] studied a single server retrial queue with unreliable server and set-up duration for service resumption of off server. Kuki et al. [11] discussed a retrial queue with customer conflicts and server breakdowns. Atencia and Galán-García [12] analysed the time spent within

a queueing system that incorporates breakdowns and varied retrial durations. Liu et al. [13] discussed an optimization model for a retrial queueing system involving servers that are not consistently reliable due to incomplete coverage and a delay in restarting after a shutdown. Poongothai et al. [14] and Saravanan et al. [15] analysed a two heterogeneous unreliable retrial queue with servers prone to customer discouragement. Yiming and Guo [16] analysed the behaviour of M/M/1 queue with unreliable servers. Saravanan et al. [17] established M/M/1 retrial queue with server breakdown, optional service, balking and feedback.

Queueing theory distinguishes between two types of multi-server systems based on the similarity of their service rates. Specifically, a system is known as homogeneous if each server within the system provides service at the same rate. On the other hand, a system is termed as heterogeneous if the servers within the system have different service rates. Yang et al. [18] explored a multi-server retrial queue considering likely starting failures of servers and Bernoulli feedback. Chang et al. [19] and Ke et al. [20] analysed unreliable multi-server retrial queues subject to customer impatience with feedback. Saravanan et al. [21] discussed a multi server unreliable retrial queue involving factors like balking, reneging and synchronised vacation.

Most queueing models are only subject to serving the incoming demands from the customer end. However, for more effective performance and for boosting the system revenue, outgoing service provision from the serving end is highly appreciable during idle times. This phenomenon where the server not only provides service to the incoming demands but also makes outgoing services is known as two way communication. The servers can avail different modes of outgoing service provision like providing service through calls, emails, chats, etc. Artalejo and Phung-Duc [22] analysed a M/M/1 retrial queueing system with two-way communication between customers and the server. Phung-Duc et al. [23] analysed a retrial queue with two-way communication, applicable to call centers with call blending, for both single and multi server cases. Sakurai and Phung-Duc [24] analysed M/M/1 retrial queues with two-way communication. Further under intense situations like high congestion, slow paced retrials and immediate connections to outgoing calls. Kumar et al. [25] considered a retrial queue where a single server engages incoming calls and outgoing calls only when free, and is prone to active failures.

Utilising the available idle time effectively will have a significant impact on the financial capital from revenue

perspective. With this aim in view, a study on effective use of idle time is extremely important. This need motivates the study of outgoing services from servers' end during idle period. Further, an investigation of different types of outgoing services to achieve a significant hike in a company's revenue is due. However, in real life practice, many service systems are providing service through multiple servers. It is essential to probe the effectiveness of several types of outgoing services in multi-server retrial service systems. Moreover, systems are often subject to technical interruptions owing to reasons varying from long exposure to faulty components. Furthermore, customers can also lose their patience being exposed to long waiting in queues. This necessitates a probe into multi-server retrial queues by considering all the discussed phenomena.

However, the existing literature on multi-server retrial queues is currently limited, despite the practical need for such systems in today's technological landscape. Additionally, the few existing multi-server queueing models that incorporate two-way communication have only considered a single type of outgoing service. Moreover, real-world scenarios often require the consideration of unreliable systems with various technical fallbacks. To address these gaps, this article presents an analysis on multiple server retrial queues with two-way communication that accounts for different modes of outgoing service and varying technical fault rates.

The paper is organized into several sections. Section 2 elucidates the practical reasoning behind the conceptual model. Section 3 introduces a QBD model, establishing stability criteria and deriving stationary probabilities for diverse server states using the MGM. Evaluation metrics of the model are accomplished in Section 4. Section 5 presents the numerical analysis results on efficiency measures. Finally, Section 6 offers conclusive remarks.

## 2. PRACTICAL IMPLEMENTATION OF MODEL

Fiber optic customer care is a feasible implementation of the conceptual model as portrayed in Figure 1. Fiber optic customer care system facilitates communication services for individuals through telecommunications. Potential customers seeking services ranging from placing order for new connection to reporting issues regarding router network, cable damages, network issues, etc., from fiber optic customer care via calls to the concerned customer care.

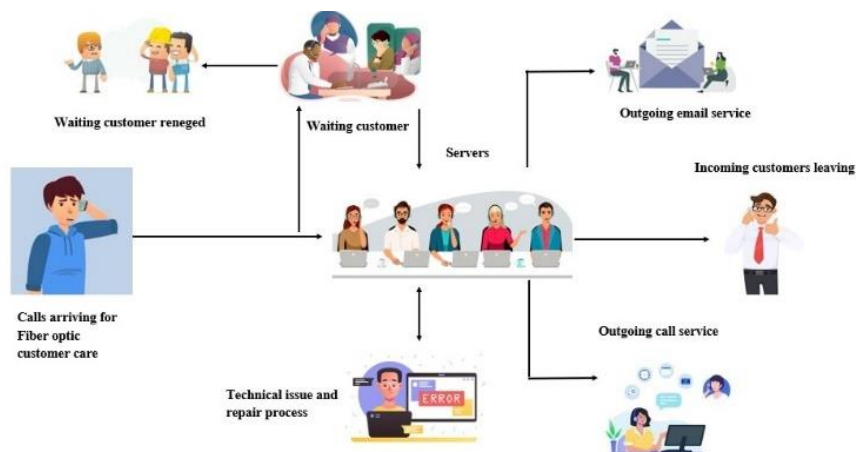


Figure 1. Fiber optic customer care centre



$$a_j^1 = \begin{bmatrix} a_{j,0}^0 & f_{j,0}^0 & & & & \\ & a_{j,0}^1 & f_{j,0}^1 & & & \\ & & \ddots & \ddots & & \\ & & & a_{j,0}^{m-1-j} & f_{j,0}^{m-1-j} & \\ & & & & & 0 \end{bmatrix},$$

$$c_k^1 = \begin{bmatrix} c_{k,0}^0 & & & & & \\ & c_{k,0}^1 & & & & \\ & & \ddots & & & \\ & & & c_{k,0}^{m-1-k} & & \\ & & & & & 0 \end{bmatrix},$$

for  $1 \leq i \leq m$ ,  $2 \leq j \leq m$  and  $1 \leq k \leq m-1$ .

$x_3$  is a  $m(m+1)(m+2)/6$  square matrix given by:

$$x_3 = \begin{bmatrix} x_1^1 & w_1^1 & & & & \\ b_2^1 & x_2^1 & w_2^1 & & & \\ & b_3^1 & x_3^1 & w_3^1 & & \\ & & \ddots & \ddots & & \\ & & & b_{m-1}^1 & x_{m-1}^1 & w_{m-1}^1 \\ & & & & b_m^1 & x_m^1 \end{bmatrix}$$

where,

$$x_i^1 = \begin{bmatrix} x_{0,i}^0 & & & & & \\ e_{0,i}^1 & x_{0,i}^1 & & & & \\ & \ddots & \ddots & & & \\ & & e_{0,i}^{m-1-i} & x_{0,i}^{m-1-i} & & \\ & & & e_{0,i}^{m-i} & x_{0,i}^{m-i} & \\ & & & & & 0 \end{bmatrix},$$

$$b_j^1 = \begin{bmatrix} b_{0,j}^0 & g_{0,j}^0 & & & & \\ & b_{0,j}^1 & g_{0,j}^1 & & & \\ & & \ddots & \ddots & & \\ & & & b_{0,j}^{m-1-j} & g_{0,j}^{m-1-j} & \\ & & & & & 0 \end{bmatrix},$$

$$w_k^1 = \begin{bmatrix} w_{0,k}^0 & & & & & \\ & w_{0,k}^1 & & & & \\ & & \ddots & & & \\ & & & w_{0,k}^{m-1-k} & & \\ & & & & & 0 \end{bmatrix},$$

for  $1 \leq i \leq m$ ,  $2 \leq j \leq m$  and  $1 \leq k \leq m-1$ .

$x_4$  is a  $m(m+1)(m^2+m-2)/24$  square matrix given by:

$$x_4 = \begin{bmatrix} A_1 & C_1 & & & & \\ B_2 & A_2 & C_2 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & & \\ & & & B_{m-2} & A_{m-2} & C_{m-2} \\ & & & & B_{m-1} & A_{m-1} \end{bmatrix}$$

where,

$$A_i = \begin{bmatrix} x_1^i & w_1^i & & & & \\ b_2^i & x_2^i & w_2^i & & & \\ & \ddots & \ddots & & & \\ & & b_{m-1-i}^i & x_{m-1-i}^i & w_{m-1-i}^i & \\ & & & b_{m-i}^i & x_{m-i}^i & \\ & & & & & 0 \end{bmatrix},$$

$$B_j = \begin{bmatrix} a_1^j & & & & & \\ & a_2^j & & & & \\ & & \ddots & & & \\ & & & a_{m-j}^j & & \\ & & & & & 0 \end{bmatrix},$$

$$C_k = \begin{bmatrix} c_1^k & & & & & \\ & c_2^k & & & & \\ & & \ddots & & & \\ & & & c_{m-1-k}^k & & \\ & & & & & 0 \end{bmatrix},$$

for  $1 \leq i \leq m-1$ ,  $2 \leq j \leq m-1$  and  $1 \leq k \leq m-2$ .

where,

$$x_j^i = \begin{bmatrix} x_{i,j}^0 & & & & & \\ e_{i,j}^1 & x_{i,j}^1 & & & & \\ & \ddots & \ddots & & & \\ & & & e_{i,j}^{m-i-j} & x_{i,j}^{m-i-j} & \\ & & & & & 0 \end{bmatrix},$$

$$b_k^i = \begin{bmatrix} b_{i,k}^0 & g_{i,k}^0 & & & & \\ & b_{i,k}^1 & g_{i,k}^1 & & & \\ & & \ddots & \ddots & & \\ & & & b_{i,k}^{m-i-k} & g_{i,k}^{m-i-k} & \\ & & & & & 0 \end{bmatrix},$$

$$w_l^i = \begin{bmatrix} w_{i,l}^0 & & & & & \\ & w_{i,l}^1 & & & & \\ & & \ddots & & & \\ & & & w_{i,l}^{m-1-i-l} & & \\ & & & & & 0 \end{bmatrix},$$

$$a_s^j = \begin{bmatrix} a_{j,s}^0 & f_{j,s}^0 & & & & \\ & a_{j,s}^1 & f_{j,s}^1 & & & \\ & & \ddots & \ddots & & \\ & & & a_{j,s}^{m-j-s} & f_{j,s}^{m-j-s} & \\ & & & & & 0 \end{bmatrix},$$

$$c_t^k = \begin{bmatrix} c_{k,t}^0 & & & & & \\ & c_{k,t}^1 & & & & \\ & & \ddots & & & \\ & & & c_{k,t}^{m-1-k-t} & & \\ & & & & & 0 \end{bmatrix},$$

for  $1 \leq i \leq m-1$ ,  $1 \leq j \leq m-i$ ,  $2 \leq k \leq m-i$ ,  $1 \leq l \leq m-1-i$ ,  $1 \leq s \leq m-j$  and  $1 \leq t \leq m-1-k$ .

The sub-matrices  $a_l$  and  $b_l$  are of order  $[m(m+1)(m+2)/6] \times [(m+1)(m+2)/2]$ :

$$a_1 = \begin{bmatrix} a_{1,0}^0 & f_{1,0}^0 & & & & \\ & a_{1,0}^1 & f_{1,0}^1 & & & \\ & & \ddots & \ddots & & \\ & & & a_{1,0}^{m-1} & f_{1,0}^{m-1} & \\ & & & & & 0 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} b_{0,1}^0 & g_{0,1}^0 & & & & \\ & b_{0,1}^1 & g_{0,1}^1 & & & \\ & & \ddots & \ddots & & \\ & & & b_{0,1}^{m-1} & g_{0,1}^{m-1} & \\ & & & & & 0 \end{bmatrix}$$

The sub-matrices  $c_l$  and  $w_l$  are of order  $[(m+1)(m+2)/2] \times [m(m+1)(m+2)/6]$ :

$$c_1 = \begin{bmatrix} c_{0,0}^0 & & & & & \\ & c_{0,0}^1 & & & & \\ & & \ddots & & & \\ & & & c_{0,0}^{m-1} & & \\ & & & & & 0 \end{bmatrix},$$

$$w_1 = \begin{bmatrix} w_{0,0}^0 & & & & & \\ & w_{0,0}^1 & & & & \\ & & \ddots & & & \\ & & & w_{0,0}^{m-1} & & \\ & & & & & 0 \end{bmatrix}$$

The sub-matrices  $a_2$  and  $b_2$  are of order  $[m(m+1)(m^2+m-2)/24] \times [m(m+1)(m+2)/6]$ :

$$a_2 = \begin{bmatrix} a_1^1 & & & \\ & a_1^2 & & \\ & & \ddots & \\ & & & a_1^{m-1} \end{bmatrix},$$

$$a_1^i = \begin{bmatrix} a_{1,i}^0 & f_{1,i}^0 & & & \\ & a_{1,i}^1 & f_{1,i}^1 & & \\ & & \ddots & \ddots & \\ & & & a_{1,i}^{m-2-i} & f_{1,i}^{m-2-i} \end{bmatrix};$$

$$b_2 = \begin{bmatrix} b_1^1 & & & \\ & b_2^1 & & \\ & & \ddots & \\ & & & b_{m-1}^1 \end{bmatrix},$$

$$b_i^1 = \begin{bmatrix} b_{i,1}^0 & g_{i,1}^0 & & & \\ & b_{i,1}^1 & g_{i,1}^1 & & \\ & & \ddots & \ddots & \\ & & & b_{i,1}^{m-2-i} & g_{i,1}^{m-2-i} \end{bmatrix};$$

$$1 \leq i \leq m-1$$

The sub-matrices  $c_2$  and  $w_2$  are of order  $[m(m+1)(m+2)/6] \times [m(m+1)(m^2+m-2)/24]$ :

$$c_2 = \begin{bmatrix} c_0^1 & & & \\ & c_0^2 & & \\ & & \ddots & \\ & & & c_0^{m-1} \end{bmatrix}, c_0^i = \begin{bmatrix} c_{0,i}^0 & & & \\ & c_{0,i}^1 & & \\ & & \ddots & \\ & & & c_{0,i}^{m-2-i} \end{bmatrix},$$

for  $1 \leq i \leq m-1$ .

$$w_2 = \begin{bmatrix} w_1^0 & & & \\ & w_2^0 & & \\ & & \ddots & \\ & & & w_{m-1}^0 \end{bmatrix},$$

$$w_i^0 = \begin{bmatrix} w_{i,0}^0 & & & \\ & w_{i,0}^1 & & \\ & & \ddots & \\ & & & w_{i,0}^{m-2-i} \end{bmatrix};$$

$$1 \leq i \leq m-1$$

where,

$$a_{p,q}^l = [a_{ij}]_{(m+1-(p+q+l)) \times (m+2-(p+q+l))},$$

$$a_{ij} = \begin{cases} p\mu_2, & i = j, 1 \leq i \leq m+1-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$b_{p,q}^l = [a_{ij}]_{(m+1-(p+q+l)) \times (m+2-(p+q+l))},$$

$$b_{ij} = \begin{cases} q\mu_3, & i = j, 1 \leq i \leq m+1-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$f_{p,q}^l = [f_{ij}]_{(m+1-(p+q+l)) \times (m+1-(p+q+l))},$$

$$f_{ij} = \begin{cases} p\alpha_2, & i = j, 1 \leq i \leq m+1-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$g_{p,q}^l = [a_{ij}]_{(m+1-(p+q+l)) \times (m+1-(p+q+l))},$$

$$g_{ij} = \begin{cases} q\alpha_3, & i = j, 1 \leq i \leq m+1-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$c_{p,q}^l = [a_{ij}]_{(m+1-(p+q+l)) \times (m-(p+q+l))},$$

$$c_{ij} = \begin{cases} \eta_1, & i = j, 1 \leq i \leq m-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$w_{p,q}^l = [a_{ij}]_{(m+1-(p+q+l)) \times (m-(p+q+l))},$$

$$w_{ij} = \begin{cases} \eta_2, & i = j, 1 \leq i \leq m-(p+q+l), \\ 0, & \text{otherwise} \end{cases},$$

$$x_{p,q}^l = [x_{ij}]_{(m+1-(p+q+l)) \times (m+1-(p+q+l))},$$

$$x_{ij} = \begin{cases} \lambda, & j = i+1, 1 \leq i \leq m-(p+q+l) \\ j\mu_1, & i = j+1, 1 \leq j \leq m-(p+q+l) \\ -[\lambda + \eta_1 + \eta_2 + (i-1)(\mu_1 + \alpha_1) + p(\mu_2 + \alpha_2) \\ \quad + q(\mu_3 + \alpha_3) + l(\beta_1 + \beta_2 + \beta_3) + n\gamma r], & i = j, 1 \leq i \leq m-(p+q+l) \\ -[\lambda + (m-(p+q+l))(\mu_1 + \alpha_1) + p(\mu_2 + \alpha_2) \\ \quad + q(\mu_3 + \alpha_3) + l(\beta_1 + \beta_2 + \beta_3) + n\gamma r], & i = j = m+1-(p+q+l) \\ 0, & \text{otherwise} \end{cases},$$

$$e_{p,q}^l = [e_{ij}]_{(m+1-(p+q+l)) \times (m+2-(p+q+l))},$$

$$h_{ij} = \begin{cases} l(\beta_1 + \beta_2 + \beta_3), & i = j, 1 \leq i \leq m+1-(p+q+l) \\ 0, & \text{otherwise} \end{cases}.$$

$y_1$  is a  $(m+1)(m+2)/2$  square matrix given by:

$$y_1 = \begin{bmatrix} y_{0,0}^0 & d_{0,0}^0 & & & \\ & y_{0,0}^1 & d_{0,0}^1 & & \\ & & \ddots & \ddots & \\ & & & y_{0,0}^{m-1} & d_{0,0}^{m-1} \\ & & & & y_{0,0}^m \end{bmatrix}$$

$y_2$  is a  $m(m+1)(m+2)/6$  square matrix given by:

$$y_2 = \begin{bmatrix} y_1^1 & & & \\ & y_2^1 & & \\ & & \ddots & \\ & & & y_m^1 \end{bmatrix},$$

where

$$y_i^1 = \begin{bmatrix} y_{i,0}^0 & d_{i,0}^0 & & & \\ & y_{i,0}^1 & d_{i,0}^1 & & \\ & & \ddots & \ddots & \\ & & & y_{i,0}^{m-1-i} & d_{i,0}^{m-1-i} \\ & & & & y_{i,0}^{m-i} \end{bmatrix}; 1 \leq i \leq m.$$

$y_3$  is a  $m(m+1)(m+2)/6$  square matrix given by:

$$y_3 = \begin{bmatrix} y_1^1 & & & \\ & y_2^1 & & \\ & & \ddots & \\ & & & y_m^1 \end{bmatrix},$$

where

$$y_j^1 = \begin{bmatrix} y_{0,j}^0 & d_{0,j}^0 & & & \\ & y_{0,j}^1 & d_{0,j}^1 & & \\ & & \ddots & \ddots & \\ & & & y_{0,j}^{m-1-j} & d_{0,j}^{m-1-j} \\ & & & & y_{0,j}^{m-j} \end{bmatrix}; 1 \leq j \leq m.$$

$y_4$  is a  $m(m+1)(m^2+m-2)/24$  square matrix given by:

$$y_4 = \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & \ddots & & \\ & & & D_{m-1} & \\ & & & & y_{m-i}^i \end{bmatrix}; D_i = \begin{bmatrix} y_1^i & & & \\ & y_2^i & & \\ & & \ddots & \\ & & & y_{m-i}^i \end{bmatrix},$$

$$y_j^i = \begin{bmatrix} y_{i,j}^0 & d_{i,j}^0 & & & \\ & y_{i,j}^1 & d_{i,j}^1 & & \\ & & \ddots & \ddots & \\ & & & y_{i,j}^{m-1-i-j} & d_{i,j}^{m-1-i-j} \\ & & & & y_{i,j}^{m-i-j} \end{bmatrix}$$

for  $1 \leq i \leq m-1$  and  $1 \leq j \leq m-1$ , where,

$$y_{p,q}^l = [y_{ij}]_{(m+1-(p+q+l)) \times (m+1-(p+q+l))},$$

$$y_{ij} = \begin{cases} \lambda, & i = j = m + 1 - (p + q + l) \\ 0, & \text{otherwise} \end{cases}$$

$$d_{p,q}^l = [d_{ij}]_{(m+1-(p+q+l)) \times (m-(p+q+l))},$$

$$d_{ij} = \begin{cases} j\alpha_1, & i = j + 1, 1 \leq j \leq m - (p + q + l) \\ 0, & \text{otherwise} \end{cases}$$

$z_1$  is a  $(m+1)(m+2)/2$  square matrix given by:

$$z_1 = \begin{bmatrix} z_{0,0}^0 & & & & \\ & z_{0,0}^1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & z_{0,0}^m \end{bmatrix}$$

$z_2$  is a  $m(m+1)(m+2)/6$  square matrix given by:

$$z_2 = \begin{bmatrix} z_1^1 & & & & \\ & z_2^1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & z_m^1 \end{bmatrix}$$

where

$$z_i^1 = \begin{bmatrix} z_{i,0}^0 & & & & \\ & z_{i,0}^1 & & & \\ & & \ddots & & \\ & & & z_{i,0}^{m-1-i} & \\ & & & & z_{i,0}^{m-i} \end{bmatrix}; 1 \leq i \leq m.$$

$z_3$  is  $m(m+1)(m+2)/6$  square matrix given by:

$$z_3 = \begin{bmatrix} z_1^1 & & & & \\ & z_2^1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & z_m^1 \end{bmatrix}$$

where

$$z_j^1 = \begin{bmatrix} z_{0,j}^0 & & & & \\ & z_{0,j}^1 & & & \\ & & \ddots & & \\ & & & z_{0,j}^{m-1-j} & \\ & & & & z_{0,j}^{m-j} \end{bmatrix}; 1 \leq i \leq m.$$

$z_4$  is a  $m(m+1)(m^2+m-2)/24$  square matrix given by:

$$z_4 = \begin{bmatrix} E_1 & & & & \\ & E_2 & & & \\ & & \ddots & & \\ & & & E_{m-1} & \\ & & & & z_{m-i}^i \end{bmatrix}; E_i = \begin{bmatrix} z_1^i & & & & \\ & z_2^i & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & z_{m-i}^i \end{bmatrix}$$

$$z_j^i = \begin{bmatrix} z_{i,j}^0 & & & & \\ & z_{i,j}^1 & & & \\ & & \ddots & & \\ & & & z_{i,j}^{m-1-i-j} & \\ & & & & z_{j,i}^{m-i-j} \end{bmatrix}$$

for  $1 \leq i \leq m-1$  and  $1 \leq j \leq m-1$ , where,

$$z_{p,q}^l = [z_{ij}]_{(m+1-(p+q+l)) \times (m+1-(p+q+l))},$$

$$z_{ij} = \begin{cases} n\gamma r, & j = i + 1, 1 \leq i \leq m - (p + q + l) \\ n\gamma \bar{r}, & i = j = m + 1 - (p + q + l) \\ 0, & \text{otherwise} \end{cases}$$

The ergodicity of the Markov process  $Q$  is established when the inequality  $xYe < xZNe$  is satisfied. Here,  $x = [x_{0,0}^{0,0}, x_{0,0}^{1,0}, \dots, x_{0,0}^{m,0}, x_{0,0}^{0,1}, x_{0,0}^{1,1}, \dots, x_{0,0}^{m-1,1}, x_{0,0}^{0,2}, \dots, x_{0,0}^{m,2}, x_{0,0}^{0,0}, x_{1,0}^{1,0}, x_{1,0}^{2,0}, \dots, x_{1,0}^{m-1,0}, x_{1,0}^{0,1}, x_{1,0}^{1,1}, \dots, x_{1,0}^{m-2,1}, x_{1,0}^{0,2}, \dots, x_{1,0}^{m-1,2}, x_{2,0}^{0,0}, x_{2,0}^{1,0}, \dots, x_{2,0}^{m-1,0}, x_{2,0}^{0,1}, x_{2,0}^{1,1}, \dots, x_{2,0}^{m-2,1}, x_{2,0}^{0,2}, \dots, x_{2,0}^{m-1,2}, x_{3,0}^{0,0}, x_{3,0}^{1,0}, \dots, x_{3,0}^{m-1,0}, x_{3,0}^{0,1}, x_{3,0}^{1,1}, \dots, x_{3,0}^{m-2,1}, x_{3,0}^{0,2}, \dots, x_{3,0}^{m-1,2}, x_{0,1}^{0,0}, x_{0,1}^{1,0}, \dots, x_{0,1}^{m-1,0}, x_{0,1}^{0,1}, x_{0,1}^{1,1}, \dots, x_{0,1}^{m-2,1}, x_{0,1}^{0,2}, \dots, x_{0,1}^{m-1,2}, x_{1,1}^{0,0}, x_{1,1}^{1,0}, \dots, x_{1,1}^{m-1,0}, x_{1,1}^{0,1}, x_{1,1}^{1,1}, \dots, x_{1,1}^{m-2,1}, x_{1,1}^{0,2}, \dots, x_{1,1}^{m-1,2}, x_{2,1}^{0,0}, x_{2,1}^{1,0}, \dots, x_{2,1}^{m-1,0}, x_{2,1}^{0,1}, x_{2,1}^{1,1}, \dots, x_{2,1}^{m-2,1}, x_{2,1}^{0,2}, \dots, x_{2,1}^{m-1,2}, x_{3,1}^{0,0}, x_{3,1}^{1,0}, \dots, x_{3,1}^{m-1,0}, x_{3,1}^{0,1}, x_{3,1}^{1,1}, \dots, x_{3,1}^{m-2,1}, x_{3,1}^{0,2}, \dots, x_{3,1}^{m-1,2}, x_{1,2}^{0,0}, x_{1,2}^{1,0}, \dots, x_{1,2}^{m-1,0}, x_{1,2}^{0,1}, x_{1,2}^{1,1}, \dots, x_{1,2}^{m-2,1}, x_{1,2}^{0,2}, \dots, x_{1,2}^{m-1,2}, x_{2,2}^{0,0}, x_{2,2}^{1,0}, \dots, x_{2,2}^{m-1,0}, x_{2,2}^{0,1}, x_{2,2}^{1,1}, \dots, x_{2,2}^{m-2,1}, x_{2,2}^{0,2}, \dots, x_{2,2}^{m-1,2}, x_{3,2}^{0,0}, x_{3,2}^{1,0}, \dots, x_{3,2}^{m-1,0}, x_{3,2}^{0,1}, x_{3,2}^{1,1}, \dots, x_{3,2}^{m-2,1}, x_{3,2}^{0,2}, \dots, x_{3,2}^{m-1,2}]$  represents the steady state distribution of  $F = X_N + Y + Z_N$ , and  $e = [1, 1, \dots, 1]^T$  is a vertical array of ones.

Solving the equation  $xF=0$ , we obtain:

$$x_{p,q}^{i,l} = \frac{1}{i!p!q!!(\beta_1 + \beta_2 + \beta_3)} \left( \frac{\lambda + N\gamma r}{\mu_1 + \alpha_1} \right)^i \left( \frac{\eta_1}{\mu_2 + \alpha_2} \right)^p \times \left( \frac{\eta_2}{\mu_3 + \alpha_3} \right)^q \left[ \frac{\alpha_1(\lambda + N\gamma r)}{\mu_1 + \alpha_1} + \frac{\alpha_2\eta_1}{\mu_2 + \alpha_2} + \frac{\alpha_3\eta_2}{\mu_3 + \alpha_3} \right] x_{0,0}^{0,0} \quad (2)$$

where,  $0 \leq i \leq m, 0 \leq p \leq m-i, 0 \leq q \leq m-(i+p), 0 \leq l \leq m-(i+p+q)$ .

Applying the normalization condition  $xe = 1, x_{0,0}^{0,0}$  is obtained as:

$$x_{0,0}^{0,0} = \left\{ \sum_{l=0}^m \sum_{i=0}^{m-l} \sum_{p=1}^{m-i} \sum_{q=1}^{m-i} \frac{1}{i!p!q!!(\beta_1 + \beta_2 + \beta_3)} \times \left( \frac{\lambda + N\gamma r}{\mu_1 + \alpha_1} \right)^i \left( \frac{\eta_1}{\mu_2 + \alpha_2} \right)^p \left( \frac{\eta_2}{\mu_3 + \alpha_3} \right)^q \times \left[ \frac{\alpha_1(\lambda + N\gamma r)}{\mu_1 + \alpha_1} + \frac{\alpha_2\eta_1}{\mu_2 + \alpha_2} + \frac{\alpha_3\eta_2}{\mu_3 + \alpha_3} \right]^{-1} \right\}^{-1} \quad (3)$$

The criteria for stability has been obtained in Neuts [26]. It has been established that the equilibrium probabilities exist only when  $xYe < xZNe$ :

$$\begin{aligned} & (\lambda - N\gamma(\bar{r} - r)) \left[ \sum_{i=0}^{m-(p+q)} \sum_{q=0}^{m-p} \sum_{p=0}^m x_{p,q}^{i,m-(p+q+i)} \right] \\ & + \alpha_1 \sum_{l=0}^{m-(p+q+i)} \sum_{q=0}^{m-(p+i)} \sum_{p=0}^{m-i} \sum_{i=1}^m i \cdot x_{p,q}^{i,l} < N\gamma r \end{aligned} \quad (4)$$

### 3.2 Matrix-geometric approach

The stationary probability vector  $\Pi = [\Pi_0, \Pi_1, \Pi_2, \dots]$  with  $\Pi_j = [P_{0,0}^{0,0}(j), P_{0,0}^{1,0}(j), \dots, P_{0,0}^{m,0}(j), P_{0,0}^{0,1}(j), P_{0,0}^{1,1}(j), \dots, P_{0,0}^{m-1,1}(j), P_{0,0}^{0,2}(j), \dots, P_{0,0}^{m,2}(j), P_{1,0}^{0,0}(j), P_{1,0}^{1,0}(j), \dots, P_{1,0}^{m-1,0}(j), P_{1,0}^{0,1}(j), P_{1,0}^{1,1}(j), \dots, P_{1,0}^{m-2,1}(j), P_{1,0}^{0,2}(j), \dots, P_{1,0}^{m-1,2}(j), P_{2,0}^{0,0}(j), P_{2,0}^{1,0}(j), \dots, P_{2,0}^{m-1,0}(j), P_{2,0}^{0,1}(j), P_{2,0}^{1,1}(j), \dots, P_{2,0}^{m-2,1}(j), P_{2,0}^{0,2}(j), \dots, P_{2,0}^{m-1,2}(j), P_{3,0}^{0,0}(j), \dots, P_{3,0}^{m,0}(j), P_{0,1}^{0,0}(j), P_{0,1}^{1,0}(j), \dots, P_{0,1}^{m-1,0}(j), P_{0,1}^{0,1}(j), P_{0,1}^{1,1}(j), \dots, P_{0,1}^{m-2,1}(j), P_{0,1}^{0,2}(j), \dots, P_{0,1}^{m-1,2}(j), P_{1,1}^{0,0}(j), \dots, P_{1,1}^{m-1,0}(j), P_{1,1}^{0,1}(j), \dots, P_{1,1}^{m-2,1}(j), P_{1,1}^{0,2}(j), \dots, P_{1,1}^{m-1,2}(j), P_{2,1}^{0,0}(j), \dots, P_{2,1}^{m-1,0}(j), P_{2,1}^{0,1}(j), \dots, P_{2,1}^{m-2,1}(j), P_{2,1}^{0,2}(j), \dots, P_{2,1}^{m-1,2}(j), P_{3,1}^{0,0}(j), \dots, P_{3,1}^{m-1,0}(j), P_{3,1}^{0,1}(j), \dots, P_{3,1}^{m-2,1}(j), P_{3,1}^{0,2}(j), \dots, P_{3,1}^{m-1,2}(j)]$ .

The array of equilibrium probabilities is obtained with the help of MGM:

$$\Pi_n = \Pi_N R^{n-N}, \quad n \geq N \quad (5)$$

where,  $R$ , the rate matrix, is the minimal nonnegative solution to  $Y + RX_N + R^2Z_N = 0$ . An approximation of  $R$  can be obtained in light of the sequence  $\{R_n\}$  as given below:

$$R_0=0 \text{ and } R_{n+1} = -YX_N^{-1} - R_n^2 Z_N X_N^{-1} \text{ for } n \geq 0$$

$\{R_n\}$  follows a monotonic pattern, allowing the evaluation of  $R$  using iterative substitution, represented as  $R = \lim_{n \rightarrow \infty} R_n$ . This approach aligns with techniques described in Neuts [26] and Latouche and Ramaswami [27].

### 3.3 Stationary distribution

$\Pi Q = 0$  results in the following:

$$\Pi_0 X_0 + \Pi_1 Z_1 = 0 \quad (6)$$

$$\Pi_{j-1} Y + \Pi_j X_j + \Pi_{j+1} Z_{j+1} = 0, \quad 1 \leq j \leq N-1 \quad (7)$$

$$\Pi_{N-1} Y + \Pi_N X_N + \Pi_{N+1} Z_N = 0 \quad (8)$$

$$\Pi_N R^{j-1-N} Y + \Pi_N R^{j-N} X_N + \Pi_N R^{j-N+1} Z_N = 0 \quad (9)$$

for  $j \geq N+1$

$$\sum_{j=0}^{\infty} \Pi_j e = 1 \quad (10)$$

By conducting algebraic operations on Eqs. (6) through (10), we derive the subsequent recursive equations:

$$\Pi_0 = \Pi_1 Z_1 (-X_0)^{-1} = \Pi_1 \chi_1 \quad (11)$$

$$\Pi_{j-1} = \Pi_j Z_j [-(\chi_{j-1} Y + X_{j-1})]^{-1} = \Pi_j \chi_j, \quad (12)$$

$2 \leq j \leq N$

$$\Pi_N (\chi_N Y + X_N + R Z_N) = 0 \quad (13)$$

where,  $\chi_1 = Z_1 (-X_0)^{-1}$  and  $\chi_j = Z_j [-(\chi_{j-1} Y + X_{j-1})]^{-1}$ ,  $2 \leq j \leq N$ .

The vector  $\Pi_N$  is determined by the help of above equations. The normalization condition leads to:

$$\sum_{j=0}^{\infty} \Pi_j e = \Pi_N [\sum_{j=1}^N \prod_{i=N}^j \chi_i + (I - R)^{-1}] e = 1 \quad (14)$$

Upon solving Eqs. (13) and (14) through Cramer's rule, we acquire  $\Pi_N$ . Additionally, the probabilities of preceding states  $[\Pi_0, \Pi_1, \dots, \Pi_{N-1}]$  are derived from Eqs. (11) and (12). Furthermore, the probabilities of subsequent states  $[\Pi_{N+1}, \Pi_{N+2}, \Pi_{N+3}, \dots]$  can be determined with the help of the formula  $\Pi_n = \Pi_N R^{n-N}$ ,  $n \geq N+1$ .

## 4. PERFORMANCE METRICS

To assess the efficiency of the conceptual model, it is essential to compute the performance metrics such as the anticipated count of the servers being in different states and the average count of orbital customers, based on steady state probability distribution.

Expected number of idle servers:

$$\begin{aligned} E[I] &= \sum_{j=0}^{\infty} \Pi_j U_1 \\ &= \sum_{j=0}^{N-1} \Pi_j U_1 + \Pi_N U_1 + \Pi_{N+1} U_1 + \Pi_{N+2} U_1 + \dots \\ &= \Pi_N [\sum_{j=1}^N \chi_j + (I - R)^{-1}] U_1 \end{aligned} \quad (15)$$

where,

$$U_1 = \begin{bmatrix} \underbrace{u_m, u_{m-1}, \dots, u_1, u_0}_{\# = m+1} \\ \underbrace{u_{m-1}, u_{m-2}, \dots, u_1, u_0}_{\# = m} \\ \dots, \underbrace{u_1, u_0}_{\# = 2}, u_0 \end{bmatrix}^T,$$

$$\text{with } u_i = \left[ \underbrace{i, i-1, \dots, 1, 0}_{\# = i+1}, \underbrace{i-1, i-2, \dots, 1, 0}_{\# = i}, \dots, \underbrace{1, 0}_{\# = 2}, 0 \right].$$

Expected number of busy incoming servers:

$$\begin{aligned} E[BI] &= \sum_{j=0}^{\infty} \Pi_j U_2 \\ &= \sum_{j=0}^{N-1} \Pi_j U_2 + \Pi_N U_2 + \Pi_{N+1} U_2 + \Pi_{N+2} U_2 + \dots \\ &= \Pi_N [\sum_{j=1}^N \chi_j + (I - R)^{-1}] U_2 \end{aligned} \quad (16)$$

where,

$$U_2 = \begin{bmatrix} \underbrace{u_m, u_{m-1}, \dots, u_1, u_0}_{\# = m+1} \\ \underbrace{u_{m-1}, u_{m-2}, \dots, u_1, u_0}_{\# = m} \\ \dots, \underbrace{u_1, u_0}_{\# = 2}, u_0 \end{bmatrix}^T,$$

$$\text{with } u_i = \left[ \underbrace{0, 1, \dots, i}_{\# = i+1}, \underbrace{0, 1, \dots, i-1}_{\# = i}, \dots, \underbrace{0, 1}_{\# = 2}, 0 \right].$$

Expected number of busy outgoing type-1 servers:

$$\begin{aligned} E[BO_1] &= \sum_{j=0}^{\infty} \Pi_j U_3 \\ &= \sum_{j=0}^{N-1} \Pi_j U_3 + \Pi_N U_3 + \Pi_{N+1} U_3 + \Pi_{N+2} U_3 + \dots \\ &= \Pi_N [\sum_{j=1}^N \chi_j + (I - R)^{-1}] U_3 \end{aligned} \quad (17)$$

where,

$$u_3 = [u_1, u_2, u_3, u_4]^T;$$

$$u_1 = \left[ \underbrace{0, 0, \dots, 0}_{\# = \frac{(m+1)(m+2)}{2}} \right],$$

$$u_2 = [u_1^1, u_2^1, \dots, u_m^1];$$

$$u_i^1 = \left[ \underbrace{i, i, \dots, i}_{\# = \frac{(m-i+1)(m-i+2)}{2}} \right],$$

$$u_3 = \left[ \underbrace{0, 0, \dots, 0}_{\# = \frac{m(m+1)(m+2)}{6}} \right],$$

$$u_4 = [u_1^1, u_2^1, \dots, u_{m-1}^1];$$

$$u_i^1 = \left[ \underbrace{i, i, \dots, i}_{\# = \frac{(m-i)(m-i+1)(m-i+2)}{6}} \right].$$

Expected number of busy outgoing type-2 servers:

$$\begin{aligned} E[BO_2] &= \sum_{j=0}^{\infty} \Pi_j U_4 \\ &= \sum_{j=0}^{N-1} \Pi_j U_4 + \Pi_N U_4 + \Pi_{N+1} U_4 + \Pi_{N+2} U_4 + \dots \\ &= \Pi_N [\sum_{j=1}^N \chi_j + (I - R)^{-1}] U_4 \end{aligned} \quad (18)$$

where,

$$u_4 = [u_1, u_2, u_3, u_4]^T,$$

$$u_1 = \left[ \underbrace{0, 0, \dots, 0}_{\# = \frac{(m+1)(m+2)}{2}} \right],$$

$$\begin{aligned}
u_2 &= \left[ \underbrace{0,0,\dots,0}_{\# \frac{m(m+1)(m+2)}{6}} \right], \\
u_3 &= [u_1^1, u_2^1, \dots, u_m^1]; \\
u_i^1 &= \left[ \underbrace{i, i, \dots, i}_{\# \frac{(m+1-i)(m+2-i)}{2}} \right], \\
u_4 &= [u_1^1, u_2^1, \dots, u_{m-1}^1, u_1^2, u_2^2, \dots, u_{m-2}^2, \dots, u_1^{m-1}]; \\
u_i^j &= \left[ \underbrace{i, i, \dots, i}_{\# \frac{(m+1-i-j)(m+2-i-j)}{2}} \right].
\end{aligned}$$

Expected number of breakdown servers:

$$\begin{aligned}
E[BD] &= \sum_{j=0}^{\infty} \Pi_j U_5 \\
&= \sum_{j=0}^{N-1} \Pi_j U_5 + \Pi_N U_5 + \Pi_{N+1} U_5 + \Pi_{N+2} U_5 + \dots \quad (19) \\
&= \Pi_N \left[ \sum_{j=1}^N \chi_j + (I - R)^{-1} \right] U_5
\end{aligned}$$

where,

$$u_5 = [u_1, u_2, \dots, u_{m+1}]^T,$$

$$u_i = \left[ \underbrace{0,0,\dots,0}_{\# = m+2-i}, \underbrace{1,1,\dots,1}_{\# = m+1-i}, \dots, \underbrace{m+1-i}_{\#1}, \underbrace{0,0,\dots,0}_{\# = m+1-i}, \underbrace{1,1,\dots,1}_{\# = m-i}, \dots, \underbrace{m-i}_{\#1}, \dots, 0 \right].$$

Expected number of customers in the orbit:

$$\begin{aligned}
E[orbit] &= \sum_{j=1}^{\infty} j \cdot \Pi_j e \\
&= \sum_{j=1}^{N-1} j \cdot \Pi_j e + N \Pi_N e + (N+1) \Pi_{N+1} e + \dots \quad (20) \\
&= \Pi_N \left[ \sum_{j=2}^N (j-1) \chi_j + N(I - R)^{-1} + R(I - R)^{-2} \right] e
\end{aligned}$$

## 5. NUMERICAL ANALYSIS

The system under consideration is a fiber optic customer care center where customer calls arrive at a rate of  $\lambda$ . Upon arrival, they are immediately served by  $m$  agents with service rate  $\mu_1$ . If all agents are busy, the customers are placed in a buffer, where they can retry for service at rate  $\gamma$ . The customers in the buffer can either continue to wait for service or abandon the system with probability  $r$  and  $1-r$  respectively.

The agents (free from incoming calls) are also responsible for making outgoing calls or emails after idle times at rate  $\eta_1$  and  $\eta_2$ , respectively. The outgoing service durations via calls and emails are exponentially distributed with rates  $\mu_2$  and  $\mu_3$  respectively. The service outlets may face technical interruptions while providing service to incoming calls, outgoing calls, and outgoing emails at rates  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  respectively. The system immediately initiates a repair process with rates  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  for incoming calls, outgoing calls, and outgoing emails, respectively.

Numerical analysis is used to validate the impact of system parameters on efficiency metrics. The effects of parameter variations on metrics like anticipated count of idle servers  $E[I]$ , servers busy with incoming calls  $E[BI]$ , servers busy with outgoing calls  $E[BO_1]$ , servers busy with outgoing emails  $E[BO_2]$ , breakdown servers  $E[BD]$ , and anticipated count of orbital customers  $E[Orbit]$  are analyzed. The choice of the parameters is subject to the satisfaction of the stability condition. The following results are obtained using MATLAB.

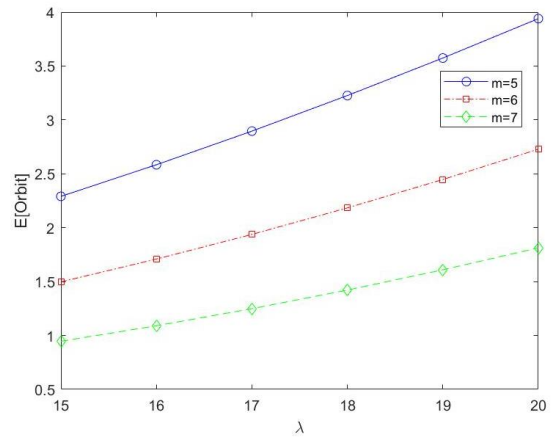


Figure 2.  $\lambda$  for various in  $m$  vs.  $E(Orbit)$

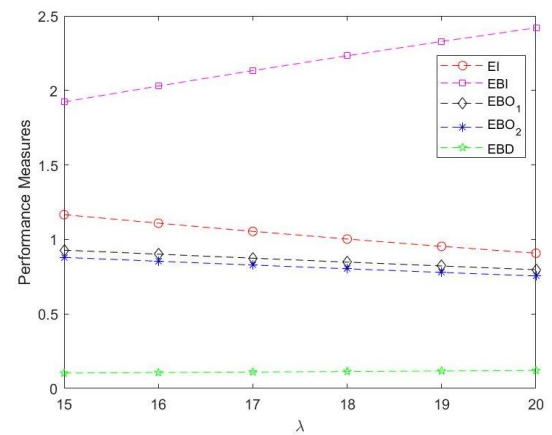


Figure 3.  $\lambda$  vs. Performance Measures

Case 1: For the following choices of parameters  $\mu_1=7, \mu_2=4, \mu_3=5, \gamma=5, r=0.7, \eta_1=6, \eta_2=7, \alpha_1=0.25, \alpha_2=0.1, \alpha_3=0.05, \beta_1=3, \beta_2=2, \beta_3=1$  and  $N=30$ , the outcomes resulting from variations in  $\lambda$  and  $m$  are portrayed in Figure 2. An escalation of the arrival rate of customer calls results in a proportional hike in the anticipated counts of orbital calls at the fiber optic customer care center. The Figure 3 demonstrates that a fixed value of  $m=5$  results in a rise in the anticipated count of agents handling incoming calls and those vulnerable to breakdown, while the expected number of idle agents, agents handling outgoing calls and agents handling outgoing emails decreases with the rise in the arrival rate of customer calls.

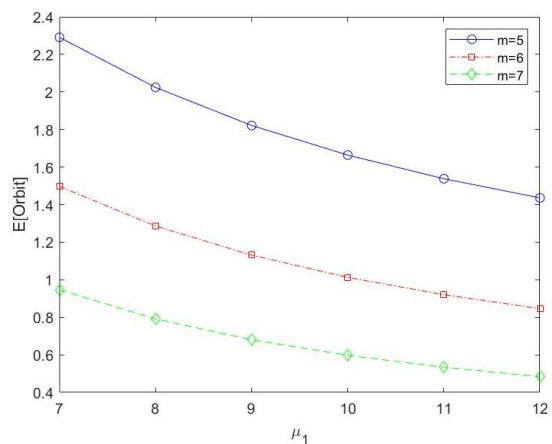


Figure 4.  $\mu_1$  for various in  $m$  vs.  $E(Orbit)$



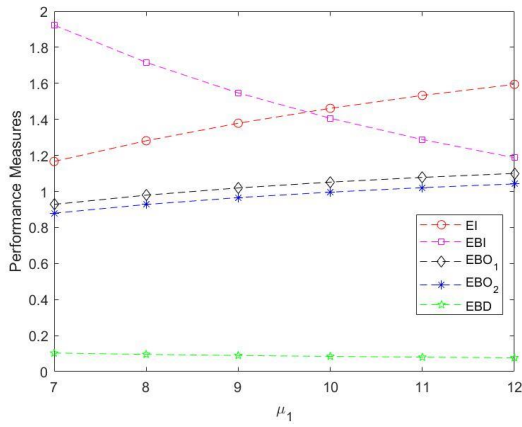


Figure 5.  $\mu_1$  vs. Performance Measures

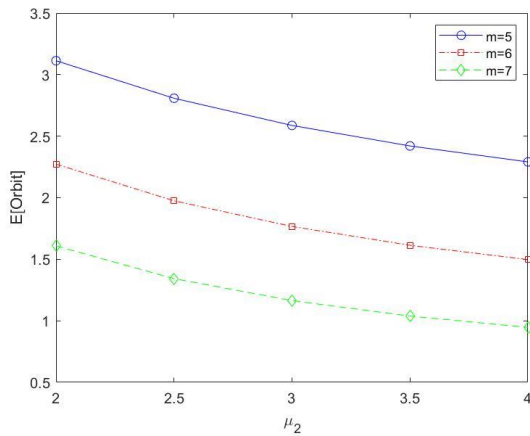


Figure 6.  $\mu_2$  for various in  $m$  vs.  $E(Orbit)$

Case 2: For  $\lambda=15$ ,  $\mu_2=4$ ,  $\mu_3=5$ ,  $\gamma=5$ ,  $r=0.7$ ,  $\eta_1=6$ ,  $\eta_2=7$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ , Figure 4 illustrates the outcomes resulting from variations in  $\mu_1$  and  $m$ . Increasing the incoming call service rate  $\mu_1$  results in a considerable decline in the anticipated count of orbital calls, regardless of the chosen value of  $m$ . Figure 5 shows that with a fixed value of  $m=5$ , a hike in  $\mu_1$  results in a decline in the anticipated count of agents handling incoming calls and those susceptible to breakdown, while the expected number of idle agents, agents handling outgoing calls, and agents handling outgoing emails increases.

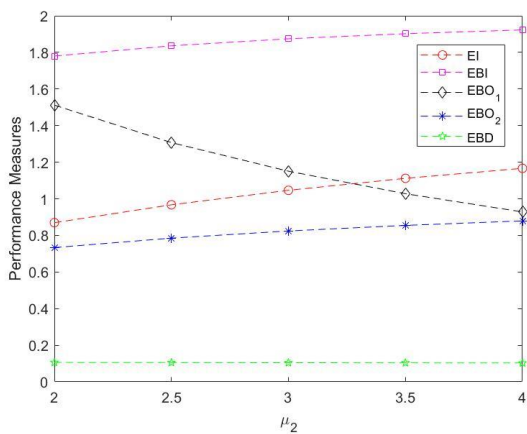


Figure 7.  $\mu_2$  vs. Performance Measures

Case 3: By choosing  $\lambda=15$ ,  $\mu_1=7$ ,  $\mu_3=5$ ,  $\gamma=5$ ,  $r=0.7$ ,  $\eta_1=6$ ,  $\eta_2=7$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ , the effects of changing  $\mu_2$  and  $m$  on system performance were investigated in Figure 6. Irrespective of the chosen value of  $m$ , increasing outgoing call service rate  $\mu_2$  resulted in a reduction of the anticipated count of fiber optic customer care orbital calls. Moreover, the increase in  $\mu_2$  with a fixed  $m=5$  led to a decline in the anticipated count of agents handling outgoing calls and those susceptible to breakdown, but a hike in the anticipated count of idle agents, agents handling incoming calls, and agents handling outgoing emails, is depicted in Figure 7.

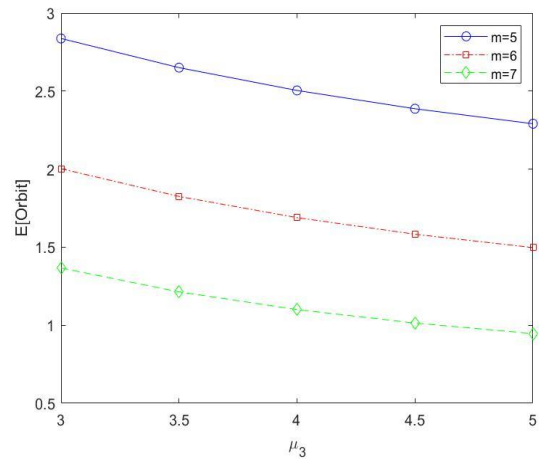


Figure 8.  $\mu_3$  for various in  $m$  vs.  $E(Orbit)$

Case 4: The study was conducted on a system with  $\lambda=15$ ,  $\mu_1=7$ ,  $\mu_2=4$ ,  $\gamma=5$ ,  $r=0.7$ ,  $\eta_1=6$ ,  $\eta_2=7$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ . The impact of varying  $\mu_3$  and  $m$  on system performance was investigated and presented in Figure 8. The results revealed that increasing outgoing email service rate  $\mu_3$  led to a reduction in the anticipated count of fiber optic customer care calls waiting in the buffer, irrespective of the chosen value of  $m$ . Additionally, with a fixed  $m=5$ , a hike in  $\mu_3$  resulted in a decline in the anticipated count of agents handling outgoing emails and those susceptible to breakdown, while increasing the expected number of idle agents, agents handling incoming calls, and agents handling outgoing calls, as illustrated in Figure 9.

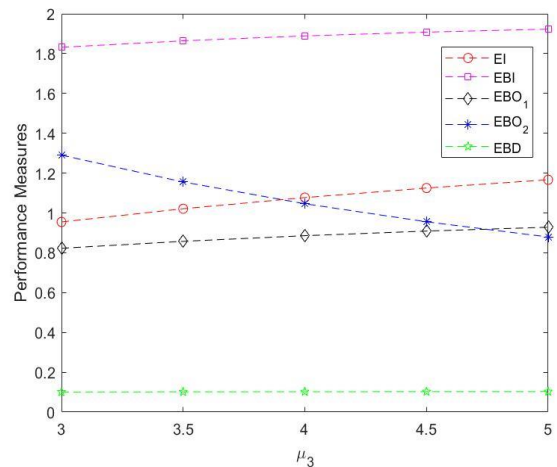


Figure 9.  $\mu_3$  vs. Performance Measures

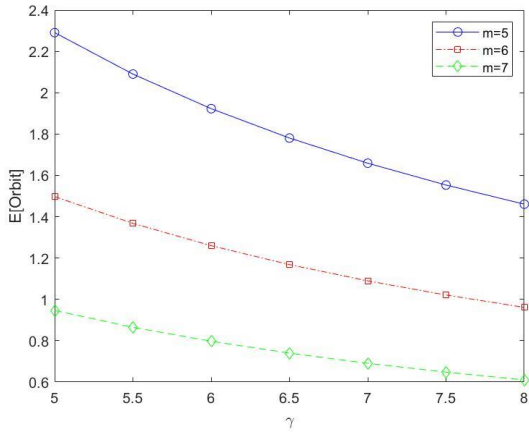


Figure 10.  $\gamma$  for various in  $m$  vs.  $E(Orbit)$

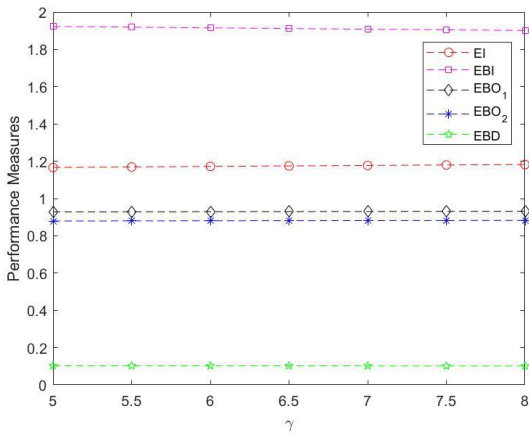


Figure 11.  $\gamma$  vs. Performance Measures

Case 5: Taking  $\lambda=15$ ,  $\mu_1=7$ ,  $\mu_2=4$ ,  $\mu_3=5$ ,  $r=0.7$ ,  $\eta_1=6$ ,  $\eta_2=7$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ . The results, presented in Figure 10, showed that increasing retrial incoming rate  $\gamma$  led to a decrease in the anticipated count of fiber optic customer care calls waiting in the buffer, regardless of the chosen  $m$ . Furthermore, when  $m=5$ , an increase in  $\gamma$  caused a reduction in the expected number of agents handling incoming calls and those at risk of breakdown, while increasing the expected number of idle agents, agents handling outgoing calls, and agents handling outgoing emails, as illustrated in Figure 11.

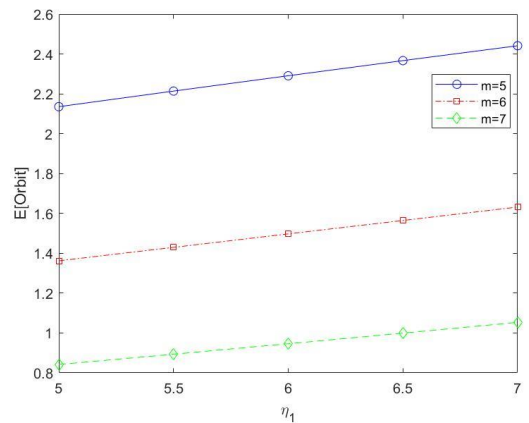


Figure 12.  $\eta_1$  for various in  $m$  vs.  $E(Orbit)$

Case 6: The system performance was evaluated for  $\lambda=15$ ,  $\mu_1=7$ ,  $\mu_2=4$ ,  $\mu_3=5$ ,  $\gamma=5$ ,  $r=0.7$ ,  $\eta_2=7$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ , and the outcomes are portrayed in Figure 12. It was noticed that increasing agents idle time before making outgoing calls rate  $\eta_1$  led to an increase in the anticipated count of fiber optic customer care calls waiting in the buffer, regardless of the chosen  $m$ . Moreover, when  $m=5$ , an increase in  $\eta_1$  caused a reduction in the expected number of idle agents, agents busy handling incoming calls and agents handling outgoing emails, while increasing the expected number of agents handling outgoing calls and breakdown, as depicted in Figure 13.

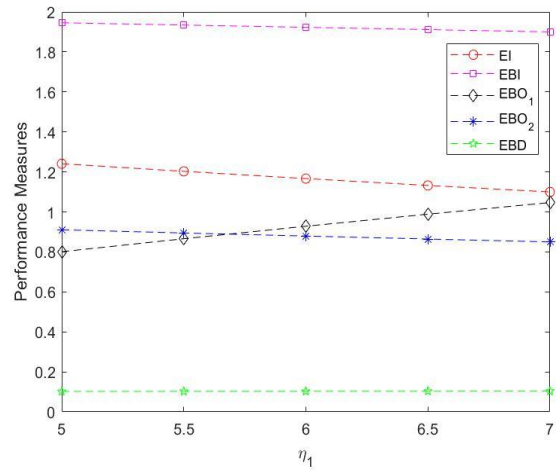


Figure 13.  $\eta_1$  vs. Performance Measures

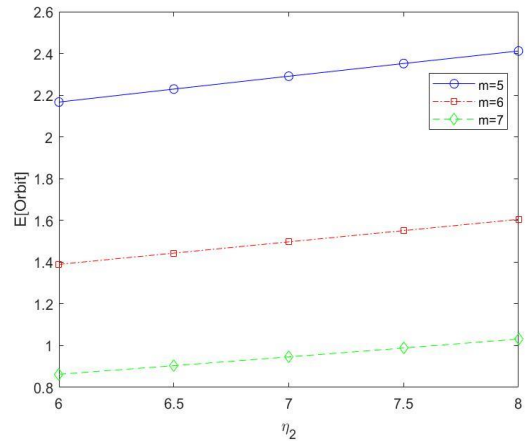
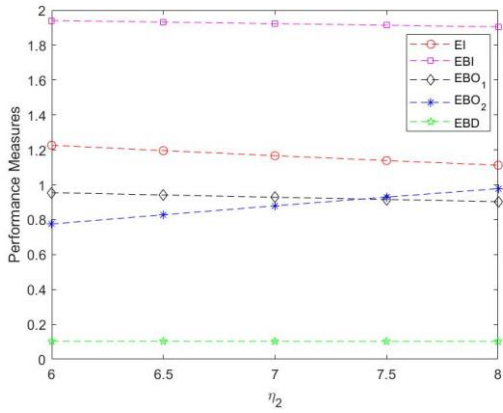


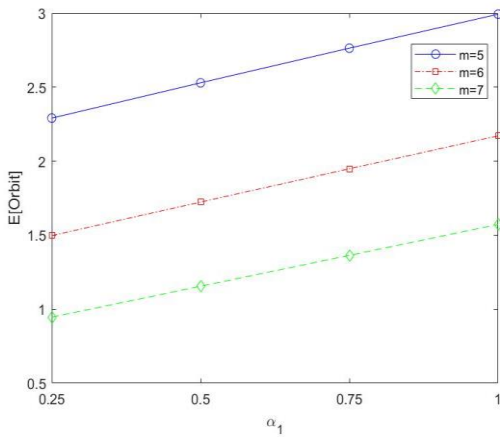
Figure 14.  $\eta_2$  for various in  $m$  vs.  $E(Orbit)$

Case 7: For the given set of parameters,  $\lambda=15$ ,  $\mu_1=7$ ,  $\mu_2=4$ ,  $\mu_3=5$ ,  $\gamma=5$ ,  $r=0.7$ ,  $\eta_1=6$ ,  $\alpha_1=0.25$ ,  $\alpha_2=0.1$ ,  $\alpha_3=0.05$ ,  $\beta_1=3$ ,  $\beta_2=2$ ,  $\beta_3=1$  and  $N=30$ , Figure 14 depicts the impact of varying agents idle time before sending outgoing emails rate  $\eta_2$  and  $m$  on the anticipated count of fiber optic customer care calls waiting in the buffer. It was observed that increasing  $\eta_2$  yields a corresponding rise in the anticipated count of calls waiting in the buffer, regardless of the chosen value of  $m$ . Furthermore, when  $m=5$ , an increase in  $\eta_2$  results in a reduction of the anticipated count of idle agents, agents occupied with incoming and outgoing calls, and agents susceptible to breakdown, while the expected number of agents handling outgoing emails increases, as presented in Figure 15.

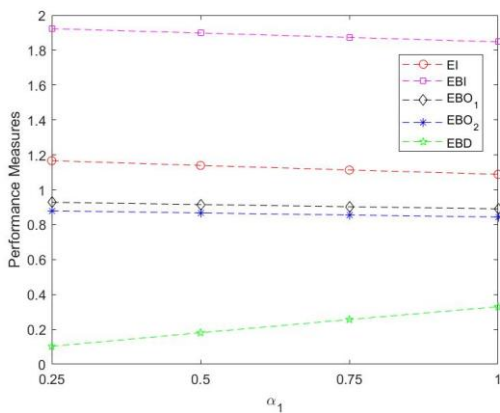


**Figure 15.**  $\eta_2$  vs. Performance Measures

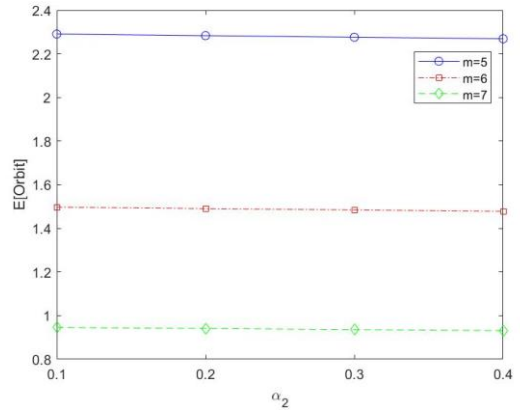
Case 8: The given parameters are  $\lambda=15, \mu_1=7, \mu_2=4, \mu_3=5, \gamma=5, r=0.7, \eta_1=6, \eta_2=7, \alpha_2=0.1, \alpha_3=0.05, \beta_1=3, \beta_2=2, \beta_3=1$  and  $N=30$ . Figure 16 shows the impact of varying rates of breakdown during incoming calls  $\alpha_1$  and  $m$  on the anticipated count of fiber optic customer care orbital calls. The results indicate that increasing  $\alpha_1$  results in a corresponding rise in the anticipated count of orbital calls, irrespective of the value of  $m$ . Moreover, when  $m=5$ , an increase in  $\alpha_1$  causes a decline in the anticipated count of idle agents, agents handling incoming calls, outgoing calls and outgoing emails, while the expected number of agents susceptible to breakdown increases, as illustrated in Figure 17.



**Figure 16.**  $\alpha_1$  for various in  $m$  vs.  $E(\text{Orbit})$

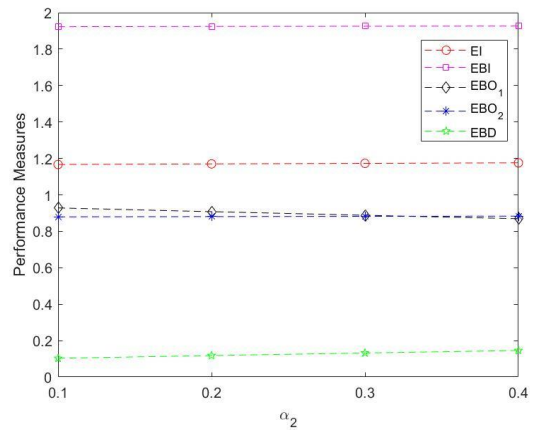


**Figure 17.**  $\alpha_1$  vs. Performance Measures

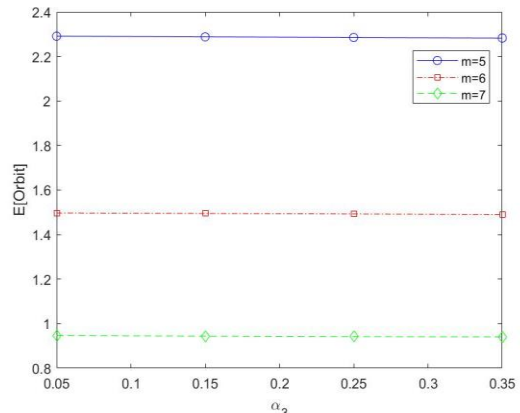


**Figure 18.**  $\alpha_2$  for various in  $m$  vs.  $E(\text{Orbit})$

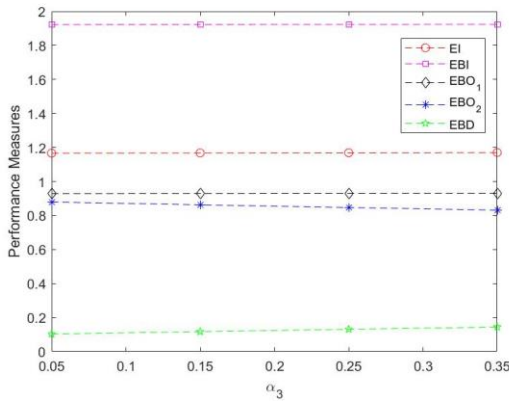
Case 9: Taking  $\lambda=15, \mu_1=7, \mu_2=4, \mu_3=5, \gamma=5, r=0.7, \eta_1=6, \eta_2=7, \alpha_1=0.25, \alpha_3=0.05, \beta_1=3, \beta_2=2, \beta_3=1$  and  $N=30$ . Figure 18 shows the impact of varying rates of breakdown during outgoing calls  $\alpha_2$  and  $m$  on the anticipated count of fiber optic customer care orbital calls. The results indicate that increasing  $\alpha_2$  results in a decline in the anticipated count of orbital calls, irrespective of the value of  $m$ . Moreover, when  $m=5$ , a rise in  $\alpha_2$  yields in a hike in the anticipated count of idle agents, agents handling incoming calls, outgoing emails, and those susceptible to breakdown, while the expected number of agents handling outgoing calls decreases, as presented in Figure 19.



**Figure 19.**  $\alpha_2$  vs. Performance Measures

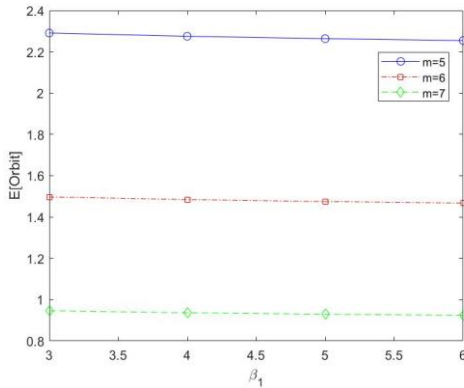


**Figure 20.**  $\alpha_3$  for various in  $m$  vs.  $E(\text{Orbit})$



**Figure 21.**  $\alpha_3$  vs. Performance Measures

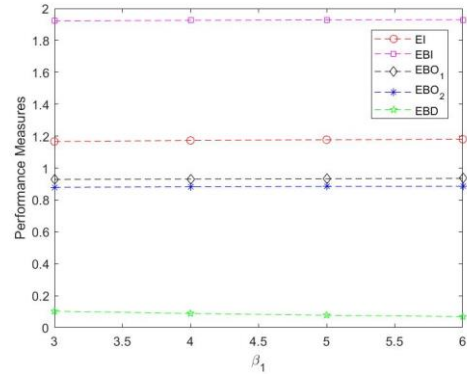
Case 10: For the given parameters  $\lambda=15, \mu_1=7, \mu_2=4, \mu_3=5, \gamma=5, r=0.7, \eta_1=6, \eta_2=7, \alpha_1=0.25, \alpha_2=0.1, \beta_1=3, \beta_2=2, \beta_3=1$  and  $N=30$ , Figure 20 depicts the effect of varying rates of breakdown during outgoing emails  $\alpha_3$  and  $m$  on the anticipated count of fiber optic customer care calls waiting in the buffer. It was observed that increasing  $\alpha_3$  led to a decline in the anticipated count of orbital calls, regardless of the chosen value of  $m$ . Furthermore, when  $m=5$ , a rise in  $\alpha_3$  yields in a hike in the anticipated count of idle agents, agents handling incoming and outgoing calls, and agents susceptible to breakdown, while the expected number of agents handling outgoing emails decreases, as presented in Figure 21.



**Figure 22.**  $\beta_1$  for various in  $m$  vs.  $E[Orbit]$

Case 11: For the given values of the parameters  $\lambda=15, \mu_1=7, \mu_2=4, \mu_3=5, \gamma=5, r=0.7, \eta_1=6, \eta_2=7, \alpha_1=0.25, \alpha_2=0.1, \alpha_3=0.05, \beta_2=2, \beta_3=1$  and  $N=30$ , Figure 22 illustrates the effect of varying rates of repair during incoming calls  $\beta_1$  and  $m$  on the anticipated count of fiber optic customer care calls waiting in the buffer. It was observed that increasing  $\beta_1$  resulted in a corresponding decline in the anticipated count of orbital calls, regardless of the chosen value of  $m$ . Moreover, when  $m=5$ , an

increase in  $\beta_1$  led to a hike in the anticipated count of idle agents, agents occupied with incoming and outgoing calls and outgoing emails, while the expected number of agents susceptible to breakdown decreased, as shown in Figure 23.



**Figure 23.**  $\beta_1$  vs. Performance Measures

Table 1 shows that as  $\lambda$  increases,  $E[BI]$  and  $E[BD]$  increase while  $E[I], E[BO_1]$  and  $E[BO_2]$  decrease. Additionally, for varying values of  $m$ , an escalation in  $\lambda$  leads to a hike in  $E[Orbit]$ . The outcomes in Table 2 imply that increasing the value of  $\mu_1$  leads to a decline in  $E[BI]$  and  $E[BD]$ , while  $E[I], E[BO_1]$  and  $E[BO_2]$  increase. Moreover, for varying values of  $m$ , a hike in  $\mu_1$  corresponds to a decline in  $E[Orbit]$ . Table 3 demonstrates that an increase in  $\mu_2$  results in a negative correlation with  $E[BO_1]$  and  $E[BD]$ , but a positive correlation with  $E[I], E[BI]$  and  $E[BO_2]$ . Additionally, there is an inverse relationship between an increase in  $\mu_2$  and  $E[Orbit]$  for varying values of  $m$ . Table 4 shows that increasing  $\mu_3$  is associated with a decrease in  $E[BO_2]$  but an increase in  $E[I], E[BI], E[BO_1]$  and  $E[BO_2]$ . Furthermore,  $E[Orbit]$  declines with a hike in  $\mu_3$  for different choices of  $m$ .

Table 5 presents that an increase in  $\gamma$  is inversely associated with  $E[BI]$  and  $E[BD]$  but positively associated with  $E[I], E[BO_1]$  and  $E[BO_2]$ . Furthermore, a hike in  $\gamma$  leads to a decline in the anticipated count  $E[Orbit]$  for varying values of  $m$ . Table 6 demonstrates that increasing  $\eta_1$  results in a decline in the expected values of  $E[I], E[BI]$  and  $E[BO_2]$ , while leading to an increase in  $E[BO_1]$  and  $E[BD]$ . Moreover, an increase in  $\eta_1$  corresponds to an increase in the expected value of  $E[Orbit]$  for diverse values of  $m$ . Table 7 displays that an increase in  $\eta_2$  is inversely associated with the expected values of  $E[I], E[BI], E[BO_1]$  and  $E[BD]$ , but positively correlated with the expected value of  $E[BO_2]$ . Additionally, a hike in  $\eta_2$  leads to an escalation in the anticipated value of  $E[Orbit]$  for varying values of  $m$ . Table 8 shows that increasing  $\alpha_1$  results in a decrease in the expected values of  $E[I], E[BI], E[BO_1]$  and  $E[BO_2]$ , while increasing the expected value of  $E[BD]$ . Moreover, an increase in  $\alpha_1$  corresponds to an increase in the expected value of  $E[Orbit]$  for different values of  $m$ .

**Table 1.** Impact of  $\lambda$  on efficiency metrics for varying server count  $m$

$\lambda$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
15	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
16	2.5841	1.7089	1.0903	1.1090	2.0298	0.9010	0.8535	0.1067
17	2.8959	1.9377	1.2487	1.0543	2.1331	0.8742	0.8281	0.1103
18	3.2261	2.1838	1.4216	1.0026	2.2326	0.8479	0.8031	0.1139
19	3.5743	2.4473	1.6095	0.9536	2.3285	0.8220	0.7786	0.1172
20	3.9399	2.7284	1.8130	0.9074	2.4206	0.7968	0.7547	0.1204

**Table 2.** Impact of  $\mu_1$  on efficiency metrics for varying server count  $m$ 

$\mu_1$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
7	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
8	2.0241	1.2862	0.7912	1.2813	1.7163	0.9793	0.9276	0.0956
9	1.8215	1.1309	0.6802	1.3786	1.5468	1.0195	0.9657	0.0895
10	1.6638	1.0127	0.5975	1.4615	1.4061	1.0518	0.9962	0.0844
11	1.5380	0.9203	0.5339	1.5328	1.2879	1.0781	1.0212	0.0801
12	1.4358	0.8463	0.4836	1.5945	1.1874	1.0999	1.0418	0.0765

**Table 3.** Impact of  $\mu_2$  on efficiency metrics for varying server count  $m$ 

$\mu_2$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
2	3.1131	2.2727	1.6084	0.8695	1.7799	1.5117	0.7334	0.1055
2.5	2.8079	1.9749	1.3423	0.9672	1.8360	1.3069	0.7850	0.1048
3	2.5871	1.7663	1.1641	1.0464	1.8745	1.1508	0.8241	0.1042
3.5	2.4205	1.6134	1.0384	1.1118	1.9022	1.0277	0.8547	0.1035
4	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029

**Table 4.** Impact of  $\mu_3$  on efficiency metrics for varying server count  $m$ 

$\mu_3$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
3	2.8370	2.0026	1.3666	0.9548	1.8318	0.8226	1.2901	0.1008
3.5	2.6502	1.8250	1.2136	1.0211	1.8644	0.8575	1.1554	0.1016
4	2.5040	1.6894	1.1004	1.0771	1.8889	0.8857	1.0461	0.1022
4.5	2.3868	1.5828	1.0139	1.1251	1.9079	0.9089	0.9555	0.1026
5	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029

**Table 5.** Impact of  $\gamma$  on efficiency metrics for varying server count  $m$ 

$\gamma$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
5	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
5.5	2.0901	1.3679	0.8651	1.1694	1.9192	0.9289	0.8798	0.1028
6	1.9226	1.2598	0.7974	1.1722	1.9154	0.9294	0.8804	0.1026
6.5	1.7807	1.1682	0.7401	1.1748	1.9118	0.9300	0.8809	0.1025
7	1.6589	1.0896	0.6908	1.1774	1.9082	0.9306	0.8814	0.1024
7.5	1.5533	1.0212	0.6479	1.1799	1.9047	0.9311	0.8820	0.1022
8	1.4606	0.9613	0.6103	1.1824	1.9014	0.9317	0.8825	0.1021

**Table 6.** Impact of  $\eta_1$  on efficiency metrics for varying server count  $m$ 

$\eta_1$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
5	2.1356	1.3623	0.8422	1.2409	1.9458	0.8009	0.9104	0.1020
5.5	2.2138	1.4298	0.8936	1.2028	1.9344	0.8658	0.8946	0.1025
6	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
6.5	2.3667	1.5647	0.9994	1.1322	1.9116	0.9885	0.8643	0.1033
7	2.4413	1.6319	1.0534	1.0997	1.9001	1.0467	0.8498	0.1037

**Table 7.** Impact of  $\eta_2$  on efficiency metrics for varying server count  $m$ 

$\eta_2$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
6	2.1670	1.3893	0.8626	1.2261	1.9410	0.9546	0.7750	0.1032
6.5	2.2293	1.4433	0.9040	1.1957	1.9320	0.9413	0.8279	0.1031
7	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
7.5	2.3517	1.5512	0.9886	1.1387	1.9140	0.9155	0.9291	0.1027
8	2.4117	1.6050	1.0317	1.1119	1.9049	0.9031	0.9776	0.1026

**Table 8.** Impact of  $\alpha_1$  on efficiency metrics for varying server count  $m$ 

$\alpha_1$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
0.25	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
0.5	2.5294	1.7238	1.1550	1.1392	1.8980	0.9153	0.8669	0.1806
0.75	2.7635	1.9486	1.3639	1.1132	1.8728	0.9027	0.8551	0.2563
1	2.9930	2.1713	1.5726	1.0885	1.8474	0.8907	0.8436	0.3298



**Table 9.** Impact of  $\alpha_2$  on efficiency metrics for varying server count  $m$

$\alpha_2$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
<b>0.1</b>	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
<b>0.2</b>	2.2834	1.4906	0.9409	1.1698	1.9242	0.9076	0.8807	0.1178
<b>0.3</b>	2.2762	1.4843	0.9359	1.1728	1.9254	0.8879	0.8820	0.1320
<b>0.4</b>	2.2694	1.4783	0.9312	1.1757	1.9265	0.8690	0.8833	0.1456

**Table 10.** Impact of  $\alpha_3$  on efficiency metrics for varying server count  $m$

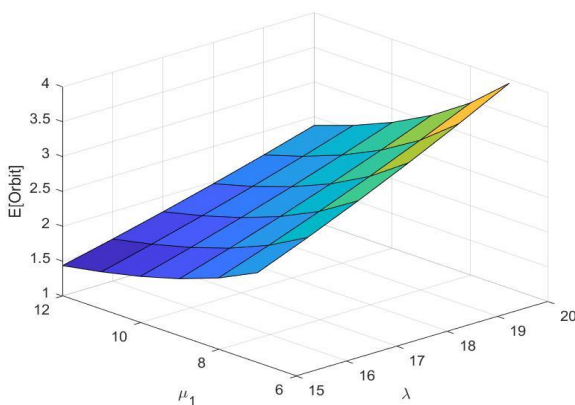
$\alpha_3$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
<b>0.05</b>	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
<b>0.15</b>	2.2880	1.4947	0.9441	1.1678	1.9235	0.9288	0.8627	0.1172
<b>0.25</b>	2.2852	1.4923	0.9421	1.1690	1.9239	0.9294	0.8468	0.1309
<b>0.35</b>	2.2826	1.4899	0.9403	1.1701	1.9243	0.9299	0.8314	0.1442

**Table 11.** Impact of  $\beta_1$  efficiency metrics for varying server count  $m$

$\beta_1$	$E[Orbit]$			$E[I]$	$E[BI]$	$E[BO_1]$	$E[BO_2]$	$E[BD]$
	$m=5$	$m=6$	$m=7$					
<b>3</b>	2.2909	1.4973	0.9461	1.1666	1.9230	0.9283	0.8792	0.1029
<b>4</b>	2.2753	1.4845	0.9366	1.1727	1.9257	0.9312	0.8820	0.0884
<b>5</b>	2.2637	1.4750	0.9295	1.1773	1.9276	0.9334	0.8842	0.0774
<b>6</b>	2.2546	1.4676	0.9240	1.1810	1.9292	0.9352	0.8858	0.0689

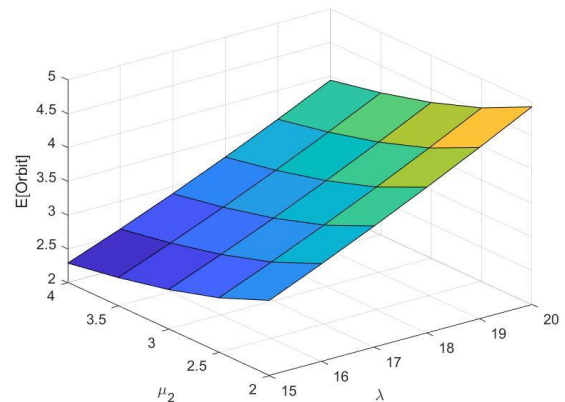
Table 9 demonstrates that an increase in  $\alpha_2$  is associated with a decrease in the expected value of  $E[Orbit]$  for varying values of  $m$ . This increase in  $\alpha_2$  also results in an escalation in the anticipated values of  $E[I]$ ,  $E[BI]$ ,  $E[BO_2]$  and  $E[BD]$ , while causing a decrease in the expected value of  $E[BO_1]$ . Table 10 shows that increasing  $\alpha_3$  results in higher expected values of  $E[I]$ ,  $E[BI]$ ,  $E[BO_1]$  and  $E[BD]$ , but lower expected value of  $E[BO_2]$ . In addition, a hike in  $\alpha_3$  causes a decline in  $E[Orbit]$  for various  $m$  values. The results in Table 11 demonstrate that increasing the values of  $\beta_1$  results in a hike in the anticipated values of  $E[I]$ ,  $E[BI]$ ,  $E[BO_1]$  and  $E[BO_2]$ , while decreasing the expected value of  $E[BD]$ . Additionally, for varying values of  $m$ , a hike in  $\beta_1$  corresponds to a decline in the expected value of  $E[Orbit]$ .

that an escalation in the arrival rate  $\lambda$  causes a rise in the anticipated count of incoming orbital calls  $E[Orbit]$ , whereas a hike in outgoing call  $\mu_2$  brings about a decrease in  $E[Orbit]$ .

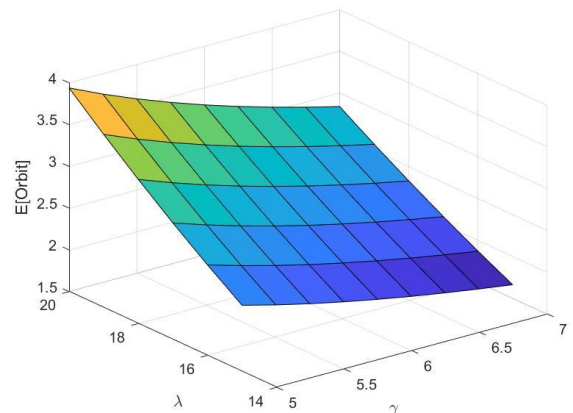


**Figure 24.**  $\lambda$  vs.  $\mu_1$  vs.  $E(Orbit)$

Three-dimensional graphical representations are utilized to explore how alterations in system parameters influence overall system performance. The anticipated count of orbital calls at the fiber optic customer care centre  $E[Orbit]$  rises with a hike in the rate of arrival  $\lambda$  and decline with an escalation in the incoming call  $\mu_1$  as portrayed in Figure 24. Figure 25 conveys

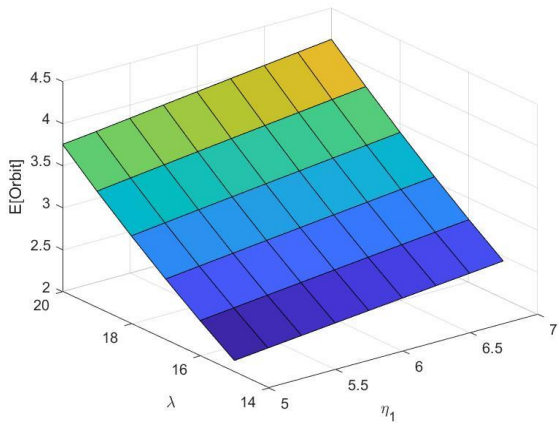


**Figure 25.**  $\lambda$  vs.  $\mu_2$  vs.  $E(Orbit)$

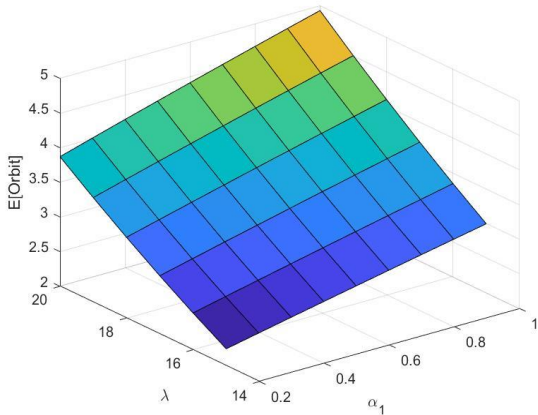


**Figure 26.**  $\gamma$  vs.  $\lambda$  vs.  $E(Orbit)$

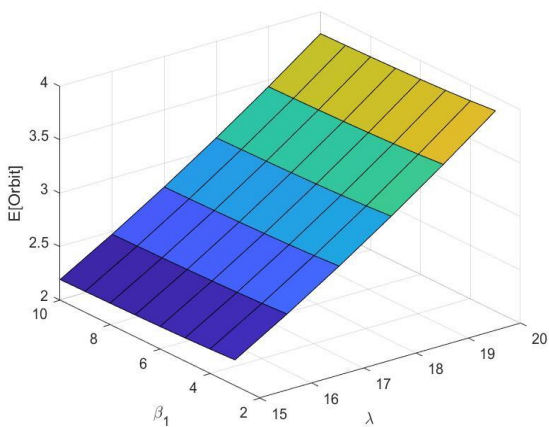
Figure 26 interprets that an escalation in the arrival rate  $\lambda$  causes a hike in the anticipated count of orbital calls  $E[Orbit]$ , whereas a hike in retrial rate  $\gamma$  brings about a decrease in  $E[Orbit]$ .



**Figure 27.**  $\eta_1$  vs.  $\lambda$  vs.  $E(Orbit)$



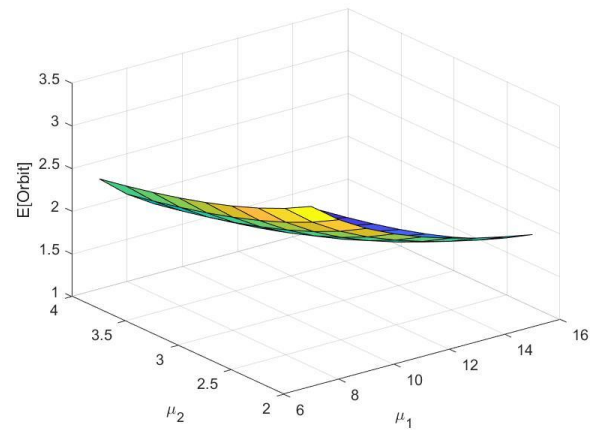
**Figure 28.**  $\alpha_1$  vs.  $\lambda$  vs.  $E(Orbit)$



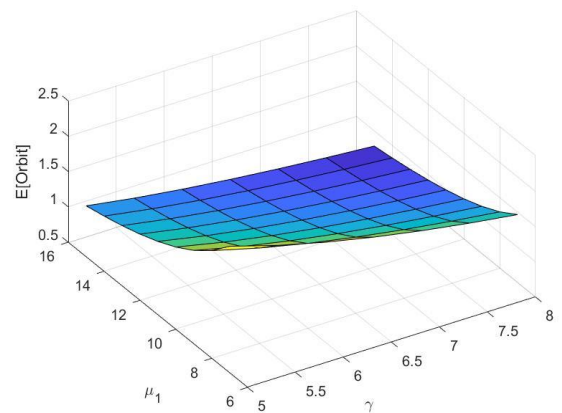
**Figure 29.**  $\lambda$  vs.  $\beta_1$  vs.  $E(Orbit)$

Rise in  $E(Orbit)$  is noted with a hike in the making outgoing call rate  $\eta_1$  and a rise in the same is witnessed with a hike in the  $\lambda$  of orbital customers is clear from Figure 27. The anticipated count of orbital calls at the fiber optic care  $E(Orbit)$  rises with a hike in  $\lambda$  and escalates with a rise in the  $\alpha_1$  as

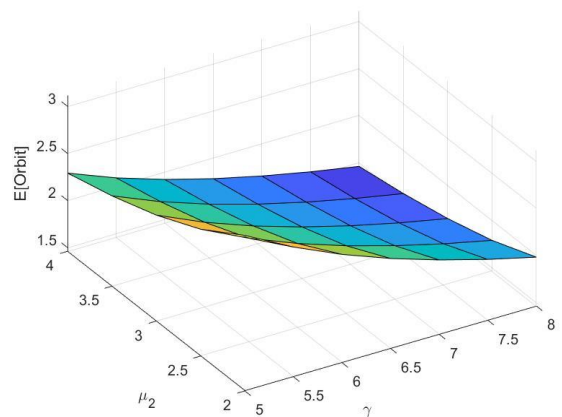
shown in Figure 28. The anticipated count of orbital calls at the fiber optic customer care centre  $E(Orbit)$  rises with a hike in  $\lambda$  and decreases with a hike in the  $\beta_1$  as portrayed in Figure 29. The anticipated count of incoming orbital calls at fiber optic customer care centre  $E(Orbit)$  reduces with a rise in incoming call service rate  $\mu_1$  and reduces with a rise in the outgoing call service rate  $\mu_2$  as shown in Figure 30.



**Figure 30.**  $\mu_1$  vs.  $\mu_2$  vs.  $E(Orbit)$



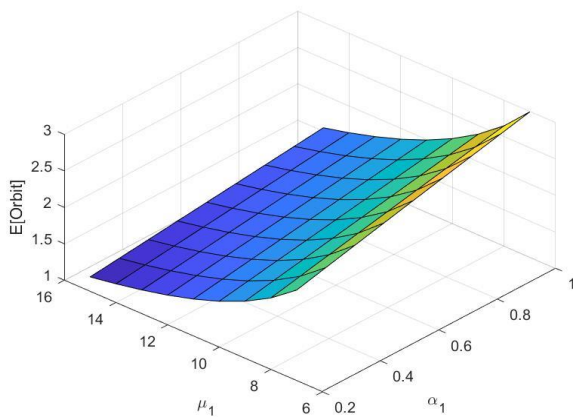
**Figure 31.**  $\gamma$  vs.  $\mu_1$  vs.  $E(Orbit)$



**Figure 32.**  $\gamma$  vs.  $\mu_2$  vs.  $E(Orbit)$

The anticipated count of orbital calls at the fiber optic customer care centre  $E(Orbit)$  reduces with a rise in retrial rate

$\gamma$  and reduces with a rise in incoming call service rate  $\mu_1$  as portrayed in Figure 31. It is evident from Figure 32 that a hike in the retrial rate  $\gamma$  brings about a decrease in the anticipated count of orbital calls  $E(Orbit)$ , whereas a hike in outgoing call service rate  $\mu_1$  brings about a decrease in  $E(Orbit)$ . The plot in Figure 33 portrays an increase in the anticipated count of orbital calls at the fiber optic customer care centre  $E(Orbit)$  for a hike in breakdown rate  $\alpha_1$  during incoming calls and decreasing trend for a hike in the incoming call service rate  $\mu_1$ .



**Figure 33.**  $\alpha_1$  vs.  $\mu_1$  vs.  $E(Orbit)$

From the numerical investigation, we infer that for enhancing the efficiency of the proposed model, an increase in the service provision rates of incoming calls, outgoing calls and emails, retrial call rates and repair rates is beneficial. In addition, minimising the duration of idle times before making outgoing services serves effective in the improvement of the system efficiency. At the same time controlling the breakdown rates has a good impact on reducing the likely chances of potential renegeing of customers. Moreover, higher arrival rates call for an increase in the number of agents.

## 6. CONCLUSIONS

The current study examines a multiple server retrial queue with unreliable servers, renegeing and two types of outgoing services. Using QBD process, we have derived the stability condition essential for the existence of steady state probabilities. The equilibrium probabilities of the system size distribution are calculated using the MGM and several efficiency metrics are evaluated to assess the performance of the proposed model. The model is illustrated with a brief example of a fiber optic customer care center. This work investigates the influence of varying system based parameters on the efficiency measures, which are visually presented through numerical and graphical illustrations. Moreover, the inference drawn from numerical investigation is also interpreted in context of the practical implementation of variations in system parameters, associated with fibre optic customer care center. This work can be further explored on retrial queues with working breakdown and working vacation.

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