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A Triple Fixed-Point Theorem for Orthogonal ℓ-Compatible Maps in Orthogonal Complete Metric Space

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ABSTRACT

Fixed-point techniques are fundamental in mathematical analysis, providing versatile tools for solving various problems across different domains. The utility of these techniques has attracted considerable interest from researchers, leading to numerous investigations and developments in this area. This article introduces the new concept of a hybrid pair of an orthogonal ℓ-compatible map on orthogonal-complete metric space. We prove some common triple-fixed-point results for such contractions. We have achieved several significant outcomes regarding triple fixed points for contraction mappings. These outcomes not only advance the theory of fixed-point theorems but also facilitate practical applications in mathematical modeling and analysis. To exhibit the potency of our approach, we provide an example that demonstrates the soundness of the new theorem premise, highlighting its relevance and applicability in real-world situations. The discoveries presented in this article have important implications for the study of integral equations. By using the triple fixed-point results established here, we can prove the existence of solutions to integral equations, which helps to solve important problems in mathematical physics, engineering, and other fields. In general, the contributions of this work expand the horizons of fixed-point theory and offer valuable insights into its applications in various areas of mathematics and beyond.

1. INTRODUCTION

In 1922, the seminal work of Banach [1] on contraction mappings led to the foundational fixed point (Fp) theorems in metric spaces (MS). These theorems have since been instrumental in various fields, extending the scope of Banach's principle to a broader context within MS. Fixed-point theorems have become a cornerstone in developing mathematical methods pivotal for tackling complex problems in applied mathematics and various scientific disciplines. The exploration of algorithms, particularly their convergence and divergence properties within optimization, and the incorporation of both ordinary and fractional differential equations, alongside integral equations, represent some of the intriguing applications of fixed-point theory.

Fixed-point theory is bifurcated into two distinct streams of inquiry: one that addresses metric spaces, which is integral to computational disciplines such as computing, computational biology, and bioinformatics, and another that focuses on topological problems, capturing the interest of topologists and theoretical computer scientists. The integration of fixed-point theory into these fields underscores its versatility and wide-ranging applications. Particularly, fractional differential equations have gained recognition for their efficacy in modeling the intricate dynamics of systems characterized by non-locality and memory effects.

Nadler's extension [2] of the Banach contraction principle to multi-valued mappings in 1969 marked a significant advancement, laying the groundwork for the exploration of Fps within the realm of Hausdorff metric spaces. Subsequent research has yielded a multitude of Fp results for various classes of multi-valued contractive mappings, as documented extensively in the literature [3-6].

The evolution of fixed-point theory continued with the introduction of the concept of coupled Fps by Guo and Lakshmikantham [7] in 1987 within the framework of partially ordered MS. This notion was further expanded to multi-valued mappings by Hussain and Alotaibi [8], with numerous authors contributing to the establishment of coupled Fp theorems across diverse MS settings.

Building on this foundation, Aydi et al. [9] in 2012 put forward the concept of coupled coincidence points for hybrid pairs of mappings, adhering to a monotone property. This was followed by the pioneering work of Berinde and Brocut [10], who presented the novel idea of tripled Fps in partially ordered MS, and Brocut [11], who explored tripled Fps for nonlinear mappings within the same context, elucidating both existence and uniqueness theorems for contractive type mappings. Amini-Haandi [12] further refined these ideas, proposing a simple and unified approach to the theories of coupled and tripled fixed points in partially ordered complete MS. For more details refer in references [13-17].

More recently, Rashwan et al. [18] broadened the scope of tripled Fps to Geraghty-type contractions in standard MSs endowed with binary relations. In the latest development, Etemad et al. [19] in 2022 validated the existence of solutions to certain problems by utilizing contraction mapping principles alongside measures of non-compactness, tripled Fp results, and the modulus of continuity.

In recent developments within the realm of fixed-point theory, the year 2017 marked a notable advancement with the work of Eshaghi Gordji et al. [20], who introduced a novel perspective on orthogonality in metric spaces, thereby providing a means to expand upon existing theorems. Expanding upon their own preliminary findings, Eshaghi Gordji and Habibi [21] furthered this line of inquiry by establishing fixed-point theorems pertinent to generalized orthogonal metric spaces. Additional contributions that delve into orthogonal concepts within this mathematical area can be found in references [22-26].

Building upon this conceptual framework, Rad et al. [27] ventured into uncharted territory by proposing the idea of orthogonal coupled fixed points within orthogonal metric spaces (O-MS). In our current study, we delve into the interplay between hybrid pairs and orthogonal ℓ-compatible mappings. We present a theorem that not only asserts the existence but also the uniqueness of tripled fixed points (TFp) in orthogonal metric spaces. This theorem we propose serves as a synthesis and expansion of numerous established results, thereby enriching the existing body of literature.

To illustrate the practical implications of our findings, we incorporate an example that underscores the effectiveness of these results. We culminate our discussion by demonstrating how the main theorem we have established can be effectively applied to ascertain the existence of non-negative solutions to integral equations.

2. PRELIMINARIES

Throughout this paper, we denote by Y, N and R a nonempty set, the set of positive integers and the set of positive real numbers, respectively.

Nadler [2] defined Fp in multi-valued maps, as follows:

Definition 2.1. An element $\varphi\in Y$ is said to be a Fp of a set-valued function $D: Y\to C_B(Y)$ iff $\varphi\in D\varphi$ [2].

The concept of coupled Fp was introduced by Hussain and Alotabi [8], in 2011.

Definition 2.2. [8] Let $\Phi: Y\times Y\to CL(Y)$ be a given map. We say that $(\varphi, \varsigma)\in Y\times Y$ is a coupled Fp of Φ iff $\varphi\in\Phi(\varphi, \varsigma)$, $\varsigma\in\Phi(\varphi, \varsigma)$.

Aydi, Abbas and Potoache [9] are presented the concept of coupled coincidence and coupled Fp of pair of maps in 2012, as follows:

Definition 2.3. Let the maps $\Phi: Y\times Y\to C_B(Y)$ and $g: Y\to Y$ be given an element $(\varphi, \varsigma)\in Y\times Y$ is called [9]:

1. a coupled coincidence point of pair $\{\Phi, g\}$ if $g\varphi\in\Phi(\varphi, \varsigma)$ and $g\varsigma\in\Phi(\varphi, \varsigma)$;
2. a coupled Fp of a pair $\{\Phi, g\}$ if $\varphi=g\varphi\in\Phi(\varphi, \varsigma)$ and $\varsigma=g\varsigma\in\Phi(\varphi, \varsigma)$.

Berinde and Borcut [10] initiated the notion of TFp and obtained a TFp theorem for single valued mapping.

Definition 2.4. Let Y be a non-empty set, a mappings $D: Y^3\to 2^Y$, and $\wr: Y\to Y$ [10]:

1. The point $(\varphi, \varsigma, \omega)\in Y^3$ is called a TFp of D if $\varphi\in D(\varphi, \varsigma, \omega)$, $\varsigma\in D(\varsigma, \varphi, \omega)$, $\omega\in D(\omega, \varsigma, \varphi)$.
2. The point $(\varphi, \varsigma, \omega)\in Y^3$ is called a TCp (tripled coincidence point) of D and $\wr$ if $\wr\varphi\in D(\varphi, \varsigma, \omega)$, $\wr\varsigma\in D(\varsigma, \varphi, \omega)$, $\wr\omega\in D(\omega, \varsigma, \varphi)$.
3. The point $(\varphi, \varsigma, \omega)\in Y^3$ is called a tripled common Fp of D and $\wr$ if $\varphi=\wr\varphi\in D(\varphi, \varsigma, \omega)$, $\varsigma=\wr\varsigma\in D(\varsigma, \varphi, \varsigma)$, $\omega=\wr\omega\in D(\omega, \varsigma, \varphi)$.

The following definition, which serves as the foundation for the rest of our work, is where we begin.

Definition 2.5. Let Y be non-void and $\perp\subseteq Y\times Y$ be a binary relation. If $\perp$ fulfills the following condition [20]:

$\exists\ \wp_0: (\forall\ \ell,\ \ell\perp\wp_0)$ or $(\forall\ \ell,\ \wp_0\perp\ell)$,

then $(Y, \perp)$ is called an O_{set}.

Definition 2.6. Let $(Y, \perp, \Re)$ be an orthogonal MS if $(Y, \perp)$ is an O_{set} and $(Y, \Re)$ is a MS [20].

Definition 2.7. Let $(Y, \perp, \Re)$ be an orthogonal MS [20].

1. Then $\varrho: Y\to Y$ is said to be orthogonally continuous in $\mu\in Y$ if for each O_{seq} (orthogonal sequence) $\{\mu_\varpi\}_{\varpi\in N}$ in Y with $\mu_\varpi\to\mu$, we have $\varrho(\mu_\varpi)\to\varrho(\mu)$. Also, ϱ is said to be orthogonal continuous on Y, if ϱ is orthogonal continuous in each $\mu\in Y$.
2. Then Y is called an orthogonally complete if every Cauchy O_{seq} is convergent.
3. A function $\varrho: Y\to Y$ is called a O_{pres} (orthogonal preserving) if $\varrho(\wp)\perp\varrho(\ell)$ if $\wp\perp\ell$.

Inspired by the hybrid pair of mappings defined by Rao et al. [26], we implement a new orthogonally hybrid pair of maps and demonstrate some TFp theorems in *O*-complete MS for this contraction mapping.

3. MAIN RESULT

We modify the notion of hybrid pair of map contraction to orthogonal sets in this article. To illustrate our results, we also give some examples and application.

Now, we begin with the definition of an orthogonal ℓ-compatible.

Definition 3.1. Let $D: Y^3\to 2^Y$ be a multi-valued mapping and $\wr$ be a self-map on Y. The Hybrid pair $\{D, \wr\}$ is said to be orthogonal ℓ-compatible if $\wr(D(\varphi, \varsigma, \omega))\subseteq D(\wr\varphi, \wr\varsigma, \wr\omega)$ whenever $(\varphi, \varsigma, \omega)$ is a TCp of D and $\wr$.

Our main theorem is to present tripled Fp theorem via the orthogonal ℓ-compatible for an orthogonal complete MS.

Theorem 3.1. Let $(Y, \perp, p)$ be an orthogonal complete MS and let $D: Y^3\to C_B(Y)$ and $\wr: Y\to Y$ maps with D is O_{pres} satisfying:

(3.1) $H(D(\varphi, \varsigma, \omega), D(\imath, \jmath, \ell))\leq\intercal p(\wr\varphi, \wr\imath)+\Re p(\wr\varsigma, \wr\jmath)+kp(\wr\omega, \wr\ell)$, $\forall\varphi, \varsigma, \omega, \imath, \jmath, \ell\in Y$ and $\intercal, \Re, k\in[0, 1)$ with $\intercal+\Re+k\leq\kappa<1$, where κ is a fixed number.

(3.2) $D(Y^3)\subseteq\wr(Y)$ and $\wr(Y)$ is a complete subspace of Y.

Then the mappings D and $\wr$ have a TCp.

Further, D and $\wr$ have a unique tripled common Fp if one of the following conditions hold:

(3.3) (a) $\{\Phi, \wr\}$ is orthogonal ℓ-compatible, $\exists\ \imath, \jmath, \ell\in Y$ s.t. $\lim_{i\to\infty}\wr^i\varphi=\imath$, $\lim_{i\to\infty}\wr^i\varsigma=\jmath$ and $\lim_{i\to\infty}\wr^i\omega=\ell$, whenever $(\varphi, \varsigma, \omega)$ is a TCp of $\{\Phi, \wr\}$ and $\wr$ is orthogonal continuous at $\imath, \jmath, \ell$.

(b) $\exists\ \imath, \jmath, \ell\in Y$ s.t. $\lim_{i\to\infty}\wr^i\imath=\varphi$, $\lim_{i\to\infty}\wr^i\jmath=\varsigma$ and $\lim_{i\to\infty}\wr^i\ell=\omega$, whenever $(\varphi, \varsigma, \omega)$ is a TCp of $\{\Phi, \wr\}$ is orthogonal continuous at φ, ς and ω.

Proof. By the definition of O_{set}, we can find $\varphi_0 \in Y$ satisfying $(\forall\ \varphi \in Y,\ \varphi \perp \varphi_0)$ or $(\forall\ \varphi \in Y,\ \varphi_0 \perp \varphi)$.

We can find $\varsigma_0 \in Y$ is satisfying $(\forall\ \varphi \in Y,\ \varphi \perp \varsigma_0)$ or $(\forall\ \varphi \in Y,\ \varsigma_0 \perp \varphi)$, and we can find $\omega_0 \in Y$ is satisfying $(\forall \varphi \in Y,\ \varphi \perp \omega_0)$ or $(\varphi \in Y,\ \omega_0 \perp \varphi)$.

It follows that $\varphi_0 \perp D(\varphi_0, \varsigma_0, \omega_0)$ or $D(\varphi_0, \varsigma_0, \omega_0) \perp \varphi_0$, $\varsigma_0 \perp D(\varsigma_0, \varphi_0, \varsigma_0)$ or $D(\varsigma_0, \varphi_0, \varsigma_0) \perp \varsigma_0$ and $\omega_0 \perp D(\omega_0, \varsigma_0, \varphi_0)$ or $D(\omega_0, \varsigma_0, \varphi_0) \perp \omega_0$.

Let $\varphi_1 = D(\varphi_0, \varsigma_0, \omega_0)$, $\varphi_2 = D^2(\varphi_0, \varsigma_0, \omega_0)$, ..., $\varphi_{i+1} = D^{i+1}(\varphi_0, \varsigma_0, \omega_0)$; $\varsigma_1 = D(\varsigma_0, \varphi_0, \varsigma_0)$, $\varsigma_2 = D^2(\varsigma_0, \varphi_0, \varsigma_0)$, ...,$\varsigma_{i+1} = D^{i+1}(\varsigma_0, \varphi_0, \varsigma_0)$, $\omega_1 = D(\omega_0, \varsigma_0, \varphi_0)$, $\omega_2 = D^2(\omega_0, \varsigma_0, \varphi_0)$, ..., $\omega_{i+1} = D^{i+1}(\omega_0, \varsigma_0, \varphi_0)$.

If $\varphi_i = \varphi_{i+1}$, $\varsigma_i = \varsigma_{i+1}$ and $\omega_i = \omega_{i+1}$ for each $I \in N \cup \{0\}$, then φ_i, ς_i, ω_i is a TFp of D. Suppose that $\varphi_i \neq \varphi_{i+1}$, $\varsigma_i \neq \varsigma_{i+1}$ and $\omega_i \neq \omega_{i+1}$ for all $i \in N \cup \{0\}$. Then p $D(\varphi_i, \varsigma_i, \omega_i)$, $D(\varphi_{i+1}, \varsigma_{i+1}, \omega_{i+1}) > 0$, p $D(\varsigma_i, \varphi_i, \varsigma_i)$, $D(\varsigma_{i+1}, \varphi_{i+1}, \varsigma_{i+1}) < 0$, p $D(\omega_i, \varsigma_i, \varphi_i)$, $D(\omega_{i+1}, \varsigma_{i+1}, \varphi_{i+1}) > 0$, for all $i \in N \cup \{0\}$. Since D is O_{pres}, we have $\varphi_i \perp \varphi_{i+1}$ or $\varphi_{i+1} \perp \varphi_i$, $\varsigma_i \perp \varsigma_{i+1}$ or $\varsigma_{i+1} \perp \varsigma_i$, $\omega_i \perp \omega_{i+1}$ or $\omega_{i+1} \perp \omega_i$, $\forall i \in N \cup \{0\}$.

Therefore, from (3.2), there exists an $O_{seq}\{\varphi_i\}$, $\{\varsigma_i\}$ and $\{\omega_i\}$ in Y s.t.

$\wr\varphi_{i+1} \in D(\varphi_i, \varsigma_i, \omega_i)$, $\wr\varsigma_{i+1} \in D(\varsigma_i, \varphi_i, \varsigma_i)$ and $\wr\omega_{i+1} \in D(\omega_i, \varsigma_i, \varphi_i)$, i=0,1,2, ……

For deduced that, we get

$$p_i^{\varphi} = p(\wr\varphi_{i-1}, \wr\varphi_i),\ p_i^{\varsigma}(\wr\varsigma_{i-1}, \wr\varsigma_i),\ p_i^{\omega} = p(\wr\omega_{i-1}, \wr\omega_i) \tag{1}$$

From (3.1), we have

$$\begin{aligned} p_2^{\varphi} &= p(\wr\varphi_1, \wr\varphi_2) \le H(D(\varphi_0, \varsigma_0, \omega_0), D(\varphi_1, \varsigma_1, \omega_1)) \\ &+ \kappa \le \mathrm{T}p(\wr\varphi_0, \wr\varphi_1) + \Re(\wr\varsigma_0, \wr\varsigma_1) + \text{ƀ}p(\wr\omega_0, \wr\omega_1) + \kappa \\ &= \mathrm{T}p_1^{\varphi} + \Re p_1^{\varsigma} + \text{ƀ}p_1^{\omega} + \kappa \end{aligned} \tag{i}$$

$$\begin{aligned} p_2^{\varsigma} &= p(\wr\varsigma_1, \varsigma_2) \le H(D(\varsigma_0, \varphi_0, \varsigma_0), D(\varsigma_1, \varphi_1, \varsigma_1)) \\ &+ \kappa \le \mathrm{T}p(\wr\varsigma_0, \wr\varsigma_1) + \Re p(\wr\varphi_0, \wr\varphi_1) \\ &+ \text{ƀ}p(\wr\varsigma_0, \wr\varsigma_1) + \kappa = \Re p_1^{\varphi} + (\mathrm{T} + \text{ƀ}) + \kappa \end{aligned} \tag{ii}$$

$$\begin{aligned} p_2^{\omega} &= p(\wr\omega_1, \wr\omega_2) \le H(D(\omega_0, \varsigma_0, \varphi_0), D(\omega_1, \varsigma_1, \varphi_1)) \\ &+ \kappa \le \mathrm{T}p(\wr\omega_0, \wr\omega_1) + \Re p(\wr\varsigma_0, \wr\varsigma_1) \\ &+ \text{ƀ}p(\wr\varphi_0, \wr\varphi_1) + \kappa = \text{ƀ}p_1^{\varphi} + \Re p_1^{\varsigma} + \mathrm{T}p_1^{\omega} \end{aligned}$$

$$\begin{aligned} p_3^{\varphi} &= p(\wr\varphi_2, \wr\varphi_3) \le H(D(\varphi_1, \varsigma_1, \omega_1), D(\varphi_2, \varsigma_2, \omega_2)) \\ &+ \kappa^2 \le \mathrm{T}p(\wr\varphi_1, \wr\varphi_2) + \Re p(\wr\varsigma_1, \wr\varsigma_2) \\ &+ \text{ƀ}p(\wr\omega_1, \wr\omega_2) + \kappa^2 = \mathrm{T}p_2^{\varphi} + \Re p_2^{\varsigma} + \text{ƀ}p_2^{\omega} + \kappa^2 \\ &\le \mathrm{T}(\mathrm{T}p_1^{\varphi} + \Re p_1^{\varsigma} + \text{ƀ}p_1^{\omega} + \kappa) + \Re(\Re p_1^{\varphi} + (\mathrm{T} + \text{ƀ})p_1^{\varsigma} + \kappa) \\ &+ \text{ƀ}(\text{ƀ}p_1^{\varphi} + \Re p_1^{\varsigma} + \mathrm{T}p_1^{\omega} + \kappa) + \kappa^2 \\ &= (\mathrm{T}^2 + \Re^2 + \text{ƀ}^2)p_1^{\varphi} + (2\mathrm{T}\Re + 2\text{ƀ}\Re)p_1^{\varsigma} + (2\mathrm{T}\text{ƀ})p_1^{\omega} + \kappa^2 + \\ &(\mathrm{T} + \Re + \text{ƀ})\kappa \\ &= (\mathrm{T}^2 + \Re^2 + \text{ƀ}^2)p_1^{\varphi} + (2\mathrm{T}\Re + 2\text{ƀ}\Re)p_1^{\varsigma} + (2\mathrm{T}\text{ƀ})p_1^{\omega} + 2\kappa^2 \end{aligned} \tag{iii}$$

$$\begin{aligned} p_3^{\varsigma} &= p(\wr\varsigma_2, \wr\varsigma_3) \le H(D(\varsigma_1, \varphi_1, \varsigma_1), D(\varsigma_2, \varphi_2, \varsigma_2)) \\ &+ \kappa^2 \le \mathrm{T}p(\wr\varsigma_1, \wr\varsigma_2) + \Re p(\wr\varphi_1, \wr\varphi_2) + \text{ƀ}p(\wr\varsigma_1, \wr\varsigma_2) + \kappa^2 \\ &= \Re p_1^{\varphi} + (\mathrm{T} + \text{ƀ})p_1^{\varsigma} + \kappa^2 \le \Re(\mathrm{T}p_1^{\varphi} + \Re p_1^{\varsigma} + \text{ƀ}p_1^{\omega} + \kappa) \\ &+ (\mathrm{T} + \text{ƀ})(\Re p_1^{\varphi} + (\mathrm{T} + \text{ƀ})p_1^{\varsigma} + \kappa) + \kappa^2 \\ &= (2\mathrm{T}\Re + 2\text{ƀ}\Re)p_1^{\varphi} + [(\mathrm{T} + \text{ƀ})^2 + \Re^2]p_1^{\varsigma} \\ &+ \Re\text{ƀ}p_1^{\omega} + \kappa^2 + (\mathrm{T} + \Re + \text{ƀ})\kappa \\ &= (2\mathrm{T}\Re + 2\text{ƀ}\Re)p_1^{\varphi} + [(\mathrm{T} + \text{ƀ})^2 + \Re^2]p_1^{\varsigma} + \Re\text{ƀ}p_1^{\omega} + 2\kappa^2 \end{aligned} \tag{iv}$$

$$\begin{aligned} p_3^{\omega} &= p(\wr\omega_2, \wr\omega_3) \le H(D(\omega_1, \varsigma_1, \varphi_1), D(\omega_2, \varsigma_2, \varphi_2)) + \kappa^2 \\ &\le \mathrm{T}p(\wr\omega_1, \wr\omega_2) + \Re p(\wr\varsigma_1, \wr\varsigma_2) + \text{ƀ}p(\wr\varphi_1, \wr\varphi_2) + \kappa^2 \\ &= \mathrm{T}p_2^{\omega} + \Re p_2^{\varsigma} + \text{ƀ}p_2^{\varphi} + \kappa^2 = \text{ƀ}p_2^{\varphi} + \Re p_2^{\varsigma} + \mathrm{T}p_2^{\omega} + \kappa^2 \\ &= \text{ƀ}(\mathrm{T}p_1^{\varphi} + \Re p_1^{\varsigma} + \Re p_1^{\varsigma} + \kappa) + \Re(\Re p_1^{\varphi} + (\mathrm{T} + \text{ƀ})p_1^{\varsigma} + \kappa] \\ &+ \mathrm{T}(\text{ƀ}p_1^{\varphi} + \Re p_1^{\varsigma} + \mathrm{T}p_1^{\omega} + \kappa) + \kappa^2 \\ &= (2\mathrm{T}\text{ƀ} + \Re^2)p_1^{\varphi} + 2(\mathrm{T}\Re + \text{ƀ}\Re)p_1^{\varsigma} + (\mathrm{T}^2 + \Re^2)p_1^{\omega} + \kappa^2 \\ &+ (\mathrm{T} + \Re + \text{ƀ})\kappa \\ &= (2\mathrm{T}\text{ƀ} + \Re^2)p_1^{\varphi} + 2(\mathrm{T}\Re + \text{ƀ}\Re)p_1^{\varsigma} + (\mathrm{T}^2 + \Re^2)p_1^{\omega} + 2\kappa^2 \end{aligned} \tag{v}$$

Let

$$F = \begin{bmatrix} \mathrm{T} & \Re & \text{ƀ} \\ \Re & \mathrm{T} + \text{ƀ} & 0 \\ \text{ƀ} & \Re & \mathrm{T} \end{bmatrix} \text{ denoted } F^2 \text{ by } \begin{bmatrix} \tau_2 & b_2 & c_2 \\ p_2 & e_2 & \wr_2 \\ g_2 & b_2 & \kappa_2 \end{bmatrix} \tag{2}$$

It is obviously $\tau_2 + b_2 + c_2 = p_2 + e_2 + \wr_2 = g_2 + b_2 + \kappa_2 = (\mathrm{T} + \Re + k)^2 \le \kappa^2 < 1$.

Now, we prove by induction:

$$F^i = \begin{bmatrix} \tau_i & b_i & c_i \\ p_i & e_i & \wr_i \\ g_i & b_i & \kappa_i \end{bmatrix} \tag{3}$$

where, $\tau_i + b_i + c_i = p_i + e_i + \wr_i = g_i + b_i + \kappa_i = (\mathrm{T} + \Re + \text{ƀ})^i \le \kappa^i < 1$.

From Eq. (3) is true for i=1, 2.

Assume Eq. (3) is true for some i. Consider

$$F^{i+1} = F^i \cdot F = \begin{bmatrix} \tau_i & b_i & c_i \\ p_i & e_i & \wr_i \\ g_i & b_i & \kappa_i \end{bmatrix} \cdot \begin{bmatrix} \mathrm{T} & \Re & \text{ƀ} \\ \Re & \mathrm{T} + \text{ƀ} & 0 \\ \text{ƀ} & \Re & \mathrm{T} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{T}\,\tau_i + \Re b_i + \text{ƀ}ci & \tau_i + (\mathrm{T} + \text{ƀ})b_i + \Re c_i & k\tau_i + \mathrm{T}\,c_i \\ \mathrm{T}\,p_i + \Re e_i + \text{ƀ}\,\wr_i & \Re p_i + (\mathrm{T} + \text{ƀ})e_i + \Re\,\wr_i & kp_i + \mathrm{T}\wr_i \\ \mathrm{T}\,g_i + \Re b_i + \text{ƀ}\kappa_i & \Re g_i + (\mathrm{T} + \text{ƀ})b_i + \Re\kappa_i & kg_i + \mathrm{T}\,\kappa_i \end{bmatrix}$$

We obtain $\tau_{i+1} + b_{i+1} + c_{i+1} = (\mathrm{T} + \Re + \text{ƀ})(\tau_i + b_i + c_i) = (\mathrm{T} + \Re + \text{ƀ})^{i+1} \le \kappa^{i+1} < 1$.

Similarly, we get $p_{i+1} + e_{i+1} + \wr_{i+1} = g_{i+1} + b_{i+1} + \kappa_{i+1} = (\text{ƀ} + \Re + \text{ƀ})^{i+1} \le \kappa^{i+1} < 1$.

Then (3) is true for all positive integer value of i.

Now from (i)− (vi) and proceeding this way, we obtain

$$\begin{bmatrix} p_{i+1}^{\varphi} \\ p_{i+1}^{\varsigma} \\ p_{i+1}^{\omega} \end{bmatrix} = \begin{bmatrix} \tau_i & b_i & c_i \\ p_i & e_i & \wr_i \\ g_i & b_i & \kappa_i \end{bmatrix} \begin{bmatrix} p_1^{\varphi} \\ p_1^{\varsigma} \\ p_1^{\omega} \end{bmatrix} + \begin{bmatrix} i\kappa^i \\ i\kappa^i \\ i\kappa^i \end{bmatrix},$$

$$\forall i = 1, 2, 3, ...$$

That is, $p_{i+1}^{\varphi} \le \tau_i p_1^{\varphi} + b_i p_1^{\varsigma} + c_i p_1^{\omega} + i\kappa^i$; $p_{i+1}^{\varsigma} \le p_i p_1^{\varphi} + e_i p_1^{\varsigma} + \wr\, p_1^{\omega} + i\kappa^i$; $p_{i+1}^{\omega} \le g_i p_1^{\varphi} + b_i p_1^{\varsigma} + \kappa_i p_1^{\omega} + i\kappa^i$, $\forall$i=1, 2, 3, ...

For j>i, we get $p(\wr\varphi_j, \wr\varphi_i) \le p(\wr\varphi_j, \wr\varphi_{j-1}) + p(\wr\varphi_{j-1}, \wr\varphi_{j-2}) + ... + p(\wr\varphi_{i+2}, \varphi_{i+1}) + p(\wr\varphi_{i+1}, \wr\varphi_i) = p_j^{\varphi} + p_{j-1}^{\varphi} + \cdots + p_{i+2}^{\varphi} + p_{i+1}^{\varphi}$, $\le \tau_{j-1}p_1^{\varphi} + b_{j-1}p_1^{\varsigma} + c_{j-1}p_1^{\omega} + (j-1)\kappa^{j-1} + \tau_{j-2}p_1^{\varphi} + b_{j-2}p_1^{\varsigma} + c_{j-2}p_1^{\omega} + (j-2)\kappa^{j-2} + ...,$ $\le \tau_{i+1}p_1^{\varphi} + b_{i+1}p_1^{\varsigma} + c_{i+1}p_1^{\omega} + (i+1)\ \kappa^{i+1} + \tau_i p_1^{\varphi} + b_i p_1^{\varsigma} + c_i p_1^{\omega} + i\kappa^i$, $\le (\tau_{j-1} + \tau_{j-2} + \ldots + \tau_{i+1} + \tau_i)\ p_1^{\varphi} + (b_{j-1} + b_{j-2} + \ldots + b_{i+1} + b_i)p_1^{\varsigma} + (c_{j-1} + c_{j-2} + c_{i+1} + c_i)p_1^{\omega} + [(j-1)\kappa^{j-1} + (j-2)\kappa^{j-2} + ... + (i+1)\kappa^{i+1} + i\ \kappa^i]$, $\le (\kappa^{j-1} + \kappa^{j-2} + \ldots + \kappa^{i+1} + \kappa^i) + (p_1^{\varphi}\ p_1^{\varsigma} + p_1^{\omega}) + \sum_{\mathrm{T}=i}^{j-1} \mathrm{T}\ \kappa^{\mathrm{T}}$, $\le \frac{\kappa^i}{1-\kappa}(p_1^{\varphi} + p_1^{\varsigma} + p_1^{\omega}) + \sum_{\mathrm{T}=i}^{j-1} \mathrm{T}\ \kappa^{\mathrm{T}} \to 0$, as $i \to \infty$, because $0 \le \kappa < 1$.

Hence $\{\wr\varphi_i\}$ is a Cauchy O_{seq}. Similarly, we can show that $\{\wr\varsigma_i\}$ and $\{\wr\omega_i\}$ are Cauchy O_{seq}.

Suppose $\wr(Y)$ is an orthogonal complete, an $O_{seq}\ \{\wr\varphi_i\}$, $\{\wr\varsigma_i\}$ and $\{\wr\omega_i\}$ are convergent to some α, β, $\gamma \in \wr(Y)$, respectively. There exists $\varphi, \varsigma, \omega \in Y$ s.t. $\alpha = \wr\varphi$, $\beta = \wr\varsigma$ and $\gamma = \wr\omega$.

Now, we get $p(D(\varphi, \varsigma, \omega), \alpha) \le p(D(\varphi, \varsigma, \omega), \wr\varphi_{i+1}) + p(\wr\varphi_{i+1}, \alpha) \le H(D(\varphi, \varsigma, \omega), D(\varphi_i, \varsigma_i, \omega_i)) + p(\wr\varphi_{i+1}, \alpha) \le \mathrm{T}p(\wr\varphi, \wr\varphi_i) + \Re p(\wr\varsigma, \wr\varsigma_i) + kp(\wr\omega, \wr\omega_i) + p(\wr\varphi_{i+1}, \alpha) = \mathrm{T}p(\alpha, \wr\varphi_i) + \Re p(\beta, \wr\varsigma_i) + kp(\gamma, \wr\omega_i) + p(\wr\varphi_{i+1}, \alpha)$.

Taking $i \to \infty$, we get $p(D(\varphi, \varsigma, \omega), \alpha) \le 0$ so that $\alpha \in D(\varphi, \varsigma, \omega)$. i.e., $\wr\varphi \in D(\varphi, \varsigma, \omega)$. Similarly, we prove $\wr\varsigma \in D(\varsigma, \varphi, \varsigma)$ and $\wr\omega \in D(\omega, \varsigma, \varphi)$.

Thus $(\varphi, \varsigma, \omega)$ is a TCp of D and $\wr$. Suppose (3.3) (a) holds.

Since $(\varphi, \varsigma, \omega)$ is a TCp of D and $\wr$, there exists $\imath, \jmath, \ell \in Y$ s.t. $\lim\limits_{i\to\infty} \wr^i\varphi=\imath$, $\lim\limits_{i\to\infty} \wr^i\varsigma=\jmath$ and $\lim\limits_{i\to\infty} \wr^i\omega=\ell$.

Since $\wr$ is an orthogonal continuous at $\imath$, $\jmath$ and ℓ, we have $\wr\imath=\imath$, $\wr\jmath=\jmath$ and $\wr\ell=\ell$.

Since $\wr\varphi\in D(\varphi, \varsigma, \omega)$, we have $\wr^2\varphi\in\wr(D(\varphi,\varsigma,\omega))\subseteq D(\wr\varphi,\wr\varsigma,\wr\omega)$.

Since $\wr\varsigma\in D(\varsigma,\varphi,\varsigma)$, we have $\wr^2\varsigma\in\wr(D(\varsigma,\varphi,\varsigma))\subseteq D(\wr\varsigma,\wr\varphi,\wr\varsigma)$.

Since $\wr\omega\in D(\omega,\varsigma,\varphi)$, we have $\wr^2\omega\in\wr(D(\omega,\varsigma,\varphi))\subseteq D(\wr\omega,\wr\varsigma,\wr\varphi)$.

Then $(\wr\varphi, \wr\varsigma, \wr\omega)$ is TCp of D and $\wr$.

Similarly, we prove that $(\wr^i\varphi, \wr^i\varsigma, \wr^i\omega)$ is a TCp D and $\wr$.

Also, it is clear $\wr^i\varphi\in D(\wr^{i-1}\varphi, \wr^{i-1}\varsigma, \wr^{i-1}\omega)$, $\wr^i\varsigma\in D(\wr^{i-1}\varsigma, \wr^{i-1}\varphi, \wr^{i-1}\varsigma)$, $\wr^i\omega\in D(\wr^{i-1}\omega, \wr^{i-1}\varsigma, \wr^{i-1}\varphi)$.

From condition (3.1), we get $p(\wr\imath, \wr\jmath, \wr\ell)\leq p(\wr\imath, \wr^i\varphi)+p(\wr^i\varphi, D(\imath,\jmath,\ell))\leq p(\wr\imath, \wr^i\varphi)+H(D(\wr^{i-1}\varphi, \wr^{i-1}\varsigma, \wr^{i-1}\omega), D(\imath, \jmath, \ell))\leq p(\wr\imath, \wr^i)+Ͳp(\wr^i\varphi, \wr\imath)+\Re p(\wr^i\varsigma, \wr\jmath)+ȶp(\wr^i\omega, \wr\ell)$.

Taking $i\to\infty$, we get $p(\wr\imath, D(\imath, \jmath, \ell))\leq 0$, $\Rightarrow \wr\imath\in D(\imath, \jmath, \ell)$.

Hence, $\imath=\wr\imath\in D(\imath, \jmath, \ell)$. Similarly, we can prove that $\jmath=\wr\jmath\in D(\jmath, \imath, \jmath)$ and $\ell=\wr\ell\in D(\ell, \jmath, \imath)$. Thus, $(\imath, \jmath, \ell)$ is a tripled common Fp of D and $\wr$. Suppose condition (3.3) (b) holds.

Since $(\varphi, \varsigma, \omega)$ is a coincidence point of $\{D, \wr\}$, there exists $\imath, \jmath, \ell\in Y$ s.t. $\lim\limits_{i\to\infty} \wr^i\imath=\varphi$, $\lim\limits_{i\to\infty} \wr^i\jmath=\varsigma$ and $\lim\limits_{i\to\infty} \wr^i\ell=\omega$.

Since $\wr$ is continuous at φ, ς, and ω, we get $\varphi=\wr\varphi$, $\varsigma=\wr\varsigma$ and $\omega=\wr\omega$.

Hence $(\varphi, \varsigma, \omega)$ is a tripled common Fp of $\{D, \wr\}$.

Now, we prove uniqueness of tripled common Fp. Assume that $(\varphi^*, \varsigma^*, \omega^*)\in Y$ is another TFp of D satisfying $(\varphi, \varsigma, \omega)\neq(\varphi^*, \varsigma^*, \omega^*)$. Then $p\,(D(\varphi, \varsigma, \omega), D(\varphi^*, \varsigma^*, \omega^*))=p(\varphi, \varphi^*)>0$, $p(D(\varsigma, \varphi, \varsigma), D(\varsigma^*, \varphi^*, \varsigma^*))=p(\varsigma, \varsigma^*)<0$, $p(D(\omega, \varsigma, \varphi)\ D(\omega^*, \varsigma^*, \varphi^*))=p(\omega, \omega^*)>0$.

Since $\{D, \wr\}$ is O_{pres}, we get $D\varphi\perp D\varphi^*$ and $\wr\varphi\perp\wr\varphi^*$ or $D\varphi^*\perp D\varphi$ and $\wr\varphi^*\perp\wr\varphi$, $D\varsigma\perp D\varsigma^*$ and $\wr\varsigma\perp\wr\varsigma^*$ or $D\varsigma^*\perp D\varsigma$ and $\wr\varsigma^*\perp\wr\varsigma$, $D\omega\perp D\omega^*$ and $\wr\omega\perp\wr\omega^*$ or $D\omega^*\perp D\omega$ and $\wr\omega^*\perp\wr\omega$.

From the condition of $H(D(\varphi, \varsigma, \omega), D(\varphi^*, \varsigma^*, \omega^*))\leq Ͳp(\wr\varphi, \wr\varphi^*)+\Re p(\wr\varsigma, \wr\varsigma^*)+kp(\wr\omega, \wr\omega^*)$, $\forall\varphi, \varsigma, \omega, \varphi^*, \varsigma^*, \omega^*\in Y$ and Ͳ, $\Re$, ȶ$\in[0, 1)$ with Ͳ$+\Re+$ȶ$\leq\kappa<1$, we have $H(D(\varphi, \varsigma, \omega), D(\varphi^*, \varsigma^*, \omega^*))=0$.

Therefore $\varphi=\varphi^*$, $\varsigma=\varsigma^*$ and $\omega=\omega^*$.

So, $\{D, \wr\}$ has a unique tripled common Fp.

The following example illustrates Theorem 3.1.

Example 3.2. Let Y= [0, 1], and define a relation $\perp$ on Y by $\varphi\perp\varsigma$, $\varsigma\perp\omega$, and $\omega\perp\varphi$, if $\varphi, \omega\geq 0$, $\varsigma\leq 0$.

Then $(Y, \perp, p)$ is an orthogonal complete MS. Define a maps $D: Y^3\to C_B(Y)$ and $\wr: Y\to Y$ defined as $D(\varphi, \varsigma, \omega)=[0, \left(\frac{1}{8}\right)\sin(\varphi)+\left(\frac{1}{4}\right)\sin(\varsigma)+\left(\frac{1}{3}\right)\sin(\omega)]$ and $\wr\varphi=\frac{7}{8}\varphi$.

$H\ (D(\varphi, \varsigma, \omega), D(\imath, \jmath, \ell))=|(\frac{1}{8}\sin\varphi+\frac{1}{4}\sin\varsigma+\frac{1}{3}\sin\omega)-(\frac{1}{8}\sin\imath+\frac{1}{4}\sin\jmath+\frac{1}{3}\sin\ell)|=|(\frac{1}{8}\sin\varphi-\frac{1}{8}\sin\imath)+(\frac{1}{4}\sin\varsigma-\frac{1}{4}\sin\jmath)+(\frac{1}{3}\sin\omega-\frac{1}{3}\sin\ell)|\leq\frac{1}{8}|\sin\varphi-\sin\imath|+\frac{1}{4}|\sin\varsigma-\sin\jmath|+\frac{1}{3}|\sin\omega-\sin\ell|\leq\frac{1}{8}|\varphi-\imath|+\frac{1}{4}|\varsigma-\jmath|+\frac{1}{3}|\omega-\ell|\leq\frac{7}{56}|\varphi-\imath|+\frac{14}{56}|\varsigma-\jmath|+\frac{56}{3X56}|\omega-\ell|\leq\frac{1}{7}|\frac{7}{8}\varphi-\frac{7}{8}\imath|+\frac{2}{7}|\frac{7}{8}\varsigma-\frac{7}{8}\jmath|+\frac{8}{21}|\frac{7}{8}\omega-\frac{7}{8}\ell|\leq\frac{1}{7}|\wr\varphi-\wr\imath|+\frac{2}{7}|\wr\varsigma-\wr\jmath|+\frac{8}{21}|\wr\omega-\wr\ell|$.

It is clear that all axioms of Theorem 3.1 are fulfilled and (0, 0, 0) is tripled common Fp of D and $\wr$.

Let $(Y, \perp)$ be an orthogonal set. Consider on the product space $Y^3\to Y$, the following condition: $(i, j, \ell)\leq(\varphi, \varsigma, \omega)$ if and only if $i\leq\varphi$, $j\leq\varsigma$, $\ell\leq\omega$, for $(i, j, \ell), (\varphi, \varsigma, \omega)\in Y^3$.

Definition 3.3. Let Y be a nonempty set and $D: Y^3\to Y$ be a map. An element $(\varphi, \varsigma, \omega)$ is called a TFp on D if $D(\varphi, \varsigma, \omega) = \varphi$, $D(\varsigma, \omega, \varphi) = \varsigma$, $D(\omega, \varphi, \varsigma) = \omega$ [28].

From, Hille and Phillips [29], Theorem 7.2.5 extended to concept of orthogonal complete MS as follows:

Definition 3.4. A function $\eta: R_+\to R_+$ is said to be superadditive if $\eta(s) + \eta(t) \leq \eta\,(s + t)$, $\forall$ s, t $\in R_+$. It is well known that every nondecreasing, convex function $\psi: R_+\to R_+$ with $\psi\,(0) = 0$ is superadditive.

Theorem 3.5. Let $(Y, \perp, d)$ be an orthogonal complete MS. Assume there exists nondecreasing functions $\psi_n: R_+\to R_+$, n = 1,2,3 such that $\psi = \psi_1 + \psi_2 + \psi_3$ is convex, $\psi\,(0) = 0$, and $\psi^n(t)\to 0$ as $n\to$ for each $t > 0$. Let $D: Y^3\to Y$ be a mapping which is nondecreasing in each of its variables and satisfying

$$\wr\,(D\,(\varphi, \varsigma, \omega), D(i, j, \ell)) \leq \psi_1(\wr(\varphi, i)) + \psi_1(\wr(\varsigma, j)) + \psi_1(\wr(\omega,\ell)) \tag{4}$$

for each $\varphi\geq i$, $\varsigma\geq j$, $\varsigma\geq j$. Suppose either

(a) D is an orthogonal continuous or;

(b) If a nondecreasing orthogonal sequence $(\varphi_k, \varsigma_k,\omega_k)\to(\varphi, \varsigma,\omega)$, then $(\varphi_k, \varsigma_k, \omega_k)\leq(\varphi, \varsigma, \omega)$, for all $k\in N$, if there exists $\varphi_0,\zeta_0,\omega_0\in Y$ with $\varphi_0\leq D(\varphi_0, \varsigma_0,\omega_0)$, $\varsigma_0\leq D(\omega_0,\varphi_0, \varsigma_0)$, $\omega_0\leq D(\varsigma_0,\omega_0,\varphi_0)$,

(c) D is *O* preserving;

(d) If for each $(\varphi, \varsigma, \omega), (i, j, \ell)\in Y^3$, then D has a unique TFp.

Proof. By the definition of O_{set}, we can find $\varphi_0\in Y$ satisfying $(\forall\ \varphi\in Y, \varphi\perp\varphi_0)$ or $(\forall\ \varphi\in Y, \varphi_0\perp\varphi)$

We can find $\varsigma_0\in Y$ is satisfying $(\forall\ \varphi\in Y, \varphi\perp\varsigma_0)$ or $(\forall\ \varphi\in Y, \varsigma_0\perp\varphi)$, and we can find $\omega_0\in Y$ is satisfying $(\forall\varphi\in Y, \varphi\perp\omega_0)$ or $(\varphi\in Y, \omega_0\perp\varphi)$.

It follows that $\varphi_0\perp D(\varphi_0, \varsigma_0, \omega_0)$ or $D(\varphi_0, \varsigma_0, \omega_0)\perp\varphi_0$, $\varsigma_0\perp D(\varsigma_0, \varphi_0, \varsigma_0)$ or $D(\varsigma_0, \varphi_0, \varsigma_0)\perp\varsigma_0$ and $\omega_0\perp D(\omega_0, \varsigma_0, \varphi_0)$ or $D(\omega_0, \varsigma_0, \varphi_0)\perp\omega_0$.

Let $\varphi_1=D(\varphi_0, \varsigma_0, \omega_0)$, $\varphi_2=D^2(\varphi_0, \varsigma_0, \omega_0)$, ..., $\varphi_{i+1}=D^{i+1}(\varphi_0, \varsigma_0, \omega_0)$; $\varsigma_1=D(\varsigma_0, \varphi_0, \varsigma_0)$, $\varsigma_2=D^2(\varsigma_0, \varphi_0, \varsigma_0)$, ...,$\varsigma_{i+1}=D^{i+1}(\varsigma_0, \varphi_0, \varsigma_0)$, $\omega_1=D(\omega_0, \varsigma_0, \varphi_0)$, $\omega_2=D^2(\omega_0, \varsigma_0, \varphi_0)$, ..., $\omega_{i+1}=D^{i+1}(\omega_0, \varsigma_0, \varphi_0)$.

If $\varphi_i=\varphi_{i+1}$, $\varsigma_i=\varsigma_{i+1}$ and $\omega_i=\omega_{i+1}$ for each $I\in N\cup\{0\}$, then φ_i, ς_i, ω_i is a TFp of D. Suppose that $\varphi_i\neq\varphi_{i+1}$, $\varsigma_i\neq\varsigma_{i+1}$ and $\omega_i\neq\omega_{i+1}$ for all $i\in N\cup\{0\}$. Then $p\ D(\varphi_i, \varsigma_i, \omega_i), D(\varphi_{i+1}, \varsigma_{i+1}, \omega_{i+1})>0$, $p\ D(\varsigma_i, \varphi_i, \varsigma_i), D(\varsigma_{i+1}, \varphi_{i+1}, \varsigma_{i+1})<0$, $p\ D(\omega_i, \varsigma_i, \varphi_i), D(\omega_{i+1}, \varsigma_{i+1}, \varphi_{i+1})>0$, for all $i\in N\cup\{0\}$. Since D is O_{pres}, we have $\varphi_i\perp\varphi_{i+1}$ or $\varphi_{i+1}\perp\varphi_i$, $\varsigma_i\perp\varsigma_{i+1}$ or $\varsigma_{i+1}\perp\varsigma_i$, $\omega_i\perp\omega_{i+1}$ or $\omega_{i+1}\perp\omega_i$, $\forall\ i\in N\cup\{0\}$.

Let (M,ϱ) be a MS which is defined by $\varrho((\varphi, \varsigma, \omega),(i,j,\ell)) = \wr(\varphi,i) + \wr(\varsigma,j) + \wr(\omega,\ell)$. Then it is straightforward to show that $(M,\perp,\varrho)$ is a orthogonal complete MS. Let $\varrho: M\to M$ be defined by

$\varrho(\varphi, \varsigma, \omega) = (D(\varphi, \varsigma, \omega),D(\varsigma, \omega, \varphi),D(\omega, \varphi, \varsigma))$.

Then from Eq. (1), we have $\wr(\varrho(\varphi,\zeta,\omega),\varrho(i,j,\ell)) = \wr(D(\varphi, \varsigma, \omega),D(i, j,\ell))+ \wr(D(\varsigma, \omega, \varphi),D(j,\ell,i)) + \wr(D(\omega, \varphi, \varsigma), D(\ell,i,j))\leq\psi_1(\wr(\varphi,i)) + \psi_2(\wr(\varsigma,j)) + \psi_3(\wr(\omega,\ell)) + \psi_3(\wr(\varphi,i)) + \psi_1(\wr(\varsigma, j))+ \psi_2(\wr(\omega, \ell)) + \psi_2(\wr(\varphi, i)) + \psi_3(\wr(\varsigma, j)) + \psi_1(\wr(\omega, \ell)) = \psi(\wr(\varphi, i)) + \psi(\wr(\varsigma, j)) + \psi(\wr(\omega,\ell))\leq\psi(\wr(\varphi, i)) + \wr(\varsigma, j) + \wr(\omega, \ell) = \psi(\wr((\varphi, \varsigma, \omega),(i, j,\ell)))$.

Since D is nondecreasing in each of its variables then ϱ is non-decreasing. From our assumptions, ϱ is either orthogonal continuous or if a nondecreasing orthogonal sequence $i_k\to i$, for $i_k,i\in M$ then $i_k\leq i$ for each $k\in N$. Also, $(\varphi_0, \varsigma_0, \omega_0)\leq\varrho(\varphi_0, \varsigma_0, \omega_0)$.

Then all the assumptions of Theorem 3.5. are satisfied. Thus, ϱ has a Fp $(\varphi_0, \varsigma_0, \omega_0)$ and so $(\varphi_0, \varsigma_0, \omega_0)$ is a TFp of D.

Now suppose that condition (c) holds. Then for each $i, j\in M$, there exists $\ell\in M$ which is comparable to φ and ζ. Thus, by Theorem 3.5., the Fp of ϱ is unique and so $(\varphi, \varsigma, \omega)$ is the unique tripled fixed point of D. Since $(\varsigma, \omega, \varphi)$ and $(\omega, \varphi, \varsigma)$ are TFp's of ϱ too then, by the uniqueness, we get $\varphi = \varsigma = \omega$.

4. APPLICATION

In this section, we use the theoretical results obtained in the previous part to clarify the existence and uniqueness of the solution for the following system:

$$
\begin{aligned}
\varphi(s) &= g(s) + \int_0^\tau G(s,\sigma)\,[I_1(\sigma,\varphi(\sigma)) + I_2(\sigma,\varsigma(\sigma)) \\
&\quad + I_3(\sigma,\omega(\sigma))]p, \\
\varsigma(s) &= g(s) + \int_0^\tau G(s,\sigma)\,[I_1(\sigma,\varsigma(\sigma)) + I_2(\sigma,\omega(\sigma)) \\
&\quad + I_3(\sigma,\varphi(\sigma))]p, \\
\omega(s) &= g(s) + \int_0^\tau G(s,\sigma)\,[I_1(\sigma,\omega(\sigma)) + I_2(\sigma,\varsigma(\sigma)) \\
&\quad + I_3(\sigma,\varphi(\sigma))]p
\end{aligned}
\tag{5}
$$

$\forall$ s$\in$ [0, τ]. Consider the following axioms:

A_1: g: [0, τ] →R and G: [0, τ] ×R→R, are an orthogonal continuous;

A_2: $I_Ꭵ$: [0, τ] ×R→R (Ꭵ=1, 2, 3) are an orthogonal continuous;

A_3: there is a constant δ>0 s.t. ∀ φ, ς∈R, $0\le I_1(\sigma,\varphi)-I^1(\sigma,\varsigma)\le\delta(\varphi-\varsigma)$, $0\le I_2(\sigma,\varphi)-I_2(\sigma,\varsigma)\le\delta(\varphi-\varsigma)$, $0\le I_3(\sigma,\varphi)-I_3(\sigma,\varsigma)\le\delta(\varphi-\varsigma)$;

A_4: $\delta^2 \max\limits_{s\in[0,\tau]}\left(\int_0^\tau G(s,\sigma)p\sigma\right)^2 \le \frac{1}{7}$.

Let ϒ=Ꞇ([0, τ], R) be the set of all orthogonal continuous real-valued functions on [0, τ] taking values in R, and let M={π∈B: π≥0}. Set p: ϒ×ϒ→B as $p(o, O)=e^s\max\limits_{s\in[0,\tau]}|o(s)-O(s)|$. It is obvious that (ϒ, ⊥, p) is an orthogonal complete MS.

Theorem 4.1. Under hypotheses (A_1) -(A_4), Eq. (5) has a solution in $ϒ^3$, where ϒ=Ꞇ([0, τ], R).

Proof. Let ϒ=Ꞇ([0, τ], R). Define a relation ⊥ on ϒ by s⊥σ iff s(α)σ(α)≥s(α) or s(α)σ(α)≥σ(α), ∀ α∈ [0, τ]. Define the operators

D: $ϒ^3 \to C_B(Y)$ and ≀: ϒ → ϒ by $D(\varphi, \varsigma, \omega)(s) = g(s) + \int_0^\tau G(s,\sigma)\,[I_1(\sigma,\varphi(\sigma)) + I_2(\sigma,\varsigma(\sigma)) + I_3(\sigma,\omega(\sigma))]p\sigma$, and $≀(\varphi_0)=\varphi_0 \forall s\in[0,\tau]$ and φ, ς, ω∈ϒ. Thus (ϒ, ⊥, p) is an orthogonal complete MS.

The triple (φ, ς, ω) is a solution of system (4) iff (φ, ς, ω) is a TFp of D. The existence of this triple follows from Theorem 3.1, since ≀ is the identity map. Therefore, it is necessary to fulfill the remaining conditions of Theorem 3.1.

For all φ, ς, ω∈ϒ with φ⊥ς, ς⊥ω, ω⊥φ and s ∈ [0, τ]. Then D is O_{pres}.

Let us consider $\theta=\frac{3}{7}$, we have H(D(φ, ς, ω), D(ı, ȷ, ℓ))=$e^s \max\limits_{s\epsilon[0,\tau]}$|D(φ, ς, ω)–D(ı, ȷ, ℓ)| =$e^s \max\limits_{s\epsilon[0,\tau]}\left|\int_0^\tau G(s,\sigma)\right.$ $[p_1(\sigma,\varphi(\sigma))+p_2(\sigma,\varsigma(\sigma))+p_3(\sigma,\omega(\sigma))]\,p\sigma-\int_0^\tau G(s,\sigma)\,[p_1(\sigma,ı(\sigma))+p_2(\sigma,ȷ(\sigma))+p_3(\sigma,\ell(\sigma))]p\sigma|$=$e^s\max\limits_{s\epsilon[0,\tau]}|\int_0^\tau G(s,\sigma)[p_1(\sigma,\varphi(\sigma))-p_1(\sigma,ı(\sigma))+p_2(\sigma,\varsigma(\sigma))-p_2(\sigma,ȷ(\sigma))+p_3(\sigma,\omega(\sigma))-p_3(\sigma,\ell(\sigma))]p\sigma|\le e^s \max\limits_{s\epsilon[0,\tau]}\left|\int_0^\tau G(s,\sigma)[\delta(\varphi(\sigma)-ı(\sigma))+\delta(\varsigma(\sigma)-ȷ(\sigma))+\delta(\omega(\sigma)-\ell(\sigma))]p\sigma\right|\le e^s \max\limits_{s\epsilon[0,\tau]}\left|\delta\int_0^\tau G(s,\sigma)[(\varphi(\sigma)-ı(\sigma))+(\varsigma(\sigma)-ȷ(\sigma))+(\omega(\sigma)-\ell(\sigma))]p\sigma\right|\le e^s(\max\limits_{s\epsilon[0,\tau]}\delta\int_0^\tau G(s,\sigma)\,p\sigma)$ $|[(\varphi(s)-ı(s))+(\varsigma(s)-ȷ(s))+(\omega(s)-\ell(s))]|\le \frac{3e^s}{7}\left|\max\limits_{s\epsilon[0,\tau]}\{(\varphi(s)-ı(s))+(\varsigma(s)-ȷ(s))+(\omega(s)-\ell(s))\}\right|$ by (A4)$\le \frac{3e^s}{7}\max\limits_{s\epsilon[0,\tau]}\{(\varphi(s)-ı(s))+(\varsigma(s)-ȷ(s))+(\omega(s)-\ell(s))\}\le\frac{3}{7}\max\{e^s\max\limits_{s\epsilon[0,\tau]}|(\varphi(s)-ı(s))|, e^s\max\limits_{s\epsilon[0,\tau]}|(\varsigma(s)-ȷ(s))|, e^s\max\limits_{s\epsilon[0,\tau]}|(\omega(s)-\ell(s))|\}\le\theta\max\{p(\varphi, ı), p(\varsigma, ȷ), p(\omega,\ell)\}$.

This means that condition (3.3) of Theorem 3.1 is fulfilled. Thus, D has a TFp (φ, ς, ω) ∈ Ꞇ([0, τ], R)×Ꞇ([0, τ], R)×Ꞇ([0, τ], R), which is a solution to Eq. (5).

5. CONCLUSIONS

The application of fixed-point methods is a cornerstone of mathematical analysis due to their broad utility across numerous areas. This has drawn the attention of numerous researchers to the potential of these techniques. Among the most notable areas of application are the analysis of algorithmic behavior, particularly the phenomena of convergence and divergence in optimization, game theory, as well as in the study of both ordinary and fractional differential equations, alongside differential and integral equations, among others.

In our study, we have formulated and proven theorems related to tripled fixed points (TFps) for mappings that are orthogonal and ℓ-compatible within the context of orthogonally complete metric spaces. We have supplemented our principal findings with illustrative examples that demonstrate their application. Additionally, we have deduced the existence of solutions for a class of complex tripled integral equations. Inspired by the pioneering efforts of Rad et al. [27], which explored the interconnections between n-tuple fixed point theorems and single fixed points, and the advances made by Roldan et al. [30] in establishing fixed point theorems within ordered metric spaces, we envision future research to extend the exploration of TFps to additional structural formations within metric spaces.

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NOMENCLATURE

Fp/Fp's	Fixed Point/Fixed Points
MS/MS's	Metric Space/Metric Spaces
TFp	Tripled Fixed Point
TCp	Tripled Coincidence Point
O_{set}	Orthogonal-set
O_{seq}	Orthogonal-sequence
O_{pres}	Orthogonal preserving