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Generalized Quadratic Functional Equation and Its Stability over Non-Archimedean Normed Space

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ABSTRACT

A functional equation is one of the most important and fascinating areas of mathematics, which involves simple algebraic manipulations and can lead to a variety of interesting results. In recent decades, numerous authors have studied different types of functional equation and its stability, such as Hyers-Ulam Stability, Hyers-Ulam-Rassias Stability, and generalized Hyers-Ulam Stability. The stability of functional equations and mixed-type functional equations has been extensively explored by numerous researchers across various spaces, yielding intriguing results primarily in the classical (Archimedean) setting. In recent years, attention has shifted towards investigating the Hyers-Ulam stability (HUS) of generalized Quadratic functional equations in non-Archimedean normed spaces. This article demonstrates the Hyers-Ulam Stability (HUS) of Quadratic functional equations. $g(3x-y)+g(x+3y)=10g(x)+10g(y), g(\upsilon x-y)+g(x+\upsilon y)=(\upsilon^2+1)g(x)+(\upsilon^2+1)g(y)$, for any integer $\upsilon \neq 0$, in NAN space by using the direct method. Also, we have given some suitable counterexamples.

1. INTRODUCTION

The inquiry into the stability of functional equations originated from a query posed by Ulam [1] in 1940 when considering the stability of group homomorphisms. The question of Ulam is as follows. "Given two groups H_1 and H_2 with the metric $d(.,.)$ on H_2 and for $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $G: H_1 \to H_2$ satisfies the inequaltiy $d(G(a,b), G(a)G(b)) < \delta$ for all $a, b \in H_1$, then there exist a homomorphism $G': H_1 \to H_2$ with $d(G(a), G'(a)) < \varepsilon$ for every $a \in H_1$?." Every solution of Cauchy functional equation $f(x+y) = f(x) + f(y)$ is said to be an additive mapping. Hyer's [2] responded to Ulam's question regarding Banach spaces in 1941. Let f be a mapping from $E \to E'$ between Banach spaces such that

$$\| f(x+y) - f(x) - f(y) \| \leq \delta$$

for every $x, y \in E$, and for some $\delta > 0$. Then there exists a single additive mapping $T: E \to E'$ such that

$$\| f(x) - T(x) \| \leq \delta \text{ for all } x \in E.$$

Furthermore, T is linear if $f(tx)$ is continuous in $t \in R$ for each fixed $x \in E$.

Rassias [3] extended Hyer's theorem in 1978 by permitting the Cauchy difference to be unbounded. Gajda [4] in 1991, answered the question of Rassias for the case $p > 1$. In 1994, Gavruta [5] gave the generalization of the Rassias theorem by replacing unbounded Cauchy difference with the general control function. Bae and Kim [6] investigated the HUS of the Q.F equation for three variables in 2001.

Moslehian and Rassias [7] showed the solution of the generalized Ulam-Hyers stability of the Cauchy functional equation in 2007. Hyer's Ulam-Rassias stability problem of the Q.F equations in NAN spaces has been proved by many researchers ([8-11]). Bae et al. [12] introduced different types of Q.F equations in 2021.

A normed space without Archimedean properties has been introduced by Hensel [13] in 1897 which is called as NAN space. Over the past thirty years, numerous physicists have found non-Archimedean space theory to be a compelling area of study for their research, especially in the realms of quantum physics, p-adic strings, and superstrings [14]. Despite the existence of non-Archimedean counterparts to many classical normed space theory, the proofs in this context are fundamentally distinct, requiring a wholly new perspective [9-11, 15, 16].

Many researchers have extensively investigated the stability of various functional equations, with particularly intriguing findings emerging in the classical (Archimedean) case. The stability problem for these functional equations has been studied in non-Archimedean spaces in recent years.

The aim of this research is to investigate the Hyers-Ulam stability (HUS) of the generalized quadratic functional equation within a NAN space.

In this current article, we prove the following Q.F equations in NAN space using direct method.

$$g(3x-y)+g(x+3y)-10g(x)-10g(y)=0.$$

$$g(vx-y)+g(x+vy)-(v^2+1)g(x)-(v^2+1)g(y)=0.$$

2. PRELIMINARIES

This section will provide an overview of some fundamental facts concerning Non-Archimedean spaces and discuss several preliminary results cited in references [16-20].

A valuation is a mapping $|\,.\,|$ from the field $\mathcal{K}$ to the non-negative real numbers, denoted as $[0,\infty)$, satisfying the following properties:

i. 0 is the unique element in $\mathcal{K}$ with a valuation of 0.
ii. The valuation of the product of two elements $|r|$ and $|s|$ is equal to the product of their valuations $|rs| = |r|.|s|$.
iii. The triangle inequality holds:

$$|r+s| \le |r|+|s|, \forall\, r,s \in \mathcal{K}.$$

A field $\mathcal{K}$ is called a valued field if $\mathcal{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are example of valuations.

Definition 2.1 A Valuation on $\mathcal{K}$ (a field) is a function $|.|: \mathcal{K} \to [0,\infty)$ which satisfies the following conditions:

(i) $|r| \ge 0$, and equality holds if and only if $r=0$;
(ii) $|rs| = |r|\,.\,|s|$ for any $r,s \in \mathcal{K}$ and
(iii) $|r+s| \le \max\{|r|,|s|\}$ (stronger triangle inequality) is said to be a non-Archimedean Valuation.

Definition 2.2 [21] Let X be a linear space over a non-Archimedean field $\mathcal{K}$ equipped with a non-trivial valuation |.|. A function norm defined from X to $\mathcal{K}$ is considered a non-Archimedean norm if it adheres to the following conditions:

(i) $\| r \| \ge 0$ and $=0$ iff $r=0$,
(ii) $\| \alpha r \| = |\alpha|\, \| r \|, \alpha \in \mathcal{K}, r \in X$,
(iii) $\|r+s\| \le \max\{\| r \|, \| s \|\}, r,s \in X$.

Then $(X, \|.\|)$ is called a NAN space.

Definition 2.3 A sequence $\{x_n\}$ in a NAN space is considered convergent if there exists an $x \in X$ such that

$$\lim_{n\to\infty} \| x_n - x \| = 0 \text{ for every } x \in X.$$

In this case, we call that $\{x_n\} \to x$ (or) $\lim_{n\to\infty} x_n = x$.

Definition 2.4 A sequence $\{x_n\}$ in a NAN space $(X, \|.\|)$ is known as the Cauchy sequence, if there exists $x \in X$ such that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0.$$

Definition 2.5 A sequence $\{x_n\}$ is Cauchy in a non-Archimedean space if and only if the sequence $\{x_{n+1}-x_n\}$ converges to zero. In the context of a complete non-Archimedean space, completeness implies that every Cauchy sequence in the space converges.

p-adic numbers stand out as significant examples of non-Archimedean spaces. A characteristic of p-adic numbers is the deviation from the Archimedean axiom, as they do not adhere to the condition: for $x, y > 0$, there exists an n in the set of natural numbers ($\mathbb{N}$) such that x is less than ny.

In this article, we introduce the generalized quadratic functional equation as follows:

$$Dg(x,y) = g(vx-y)+g(x+vy)-(v^2+1)g(x) -(v^2+1)g(y)$$

In particular,

$$Eg(x,y) = g(3x-y)+g(x+3y)-10g(x)-10g(y)$$

Consider the functional inequalities

$$\|Eg(x,y)\| \le \xi(x,y) \tag{1}$$

and

$$\|Dg(x,y)\| \le \xi(x,y) \tag{2}$$

for an upper bound function $\xi: X^2 \to [0,\infty)$.

3. STABILITY OF THE Q.F EQUATIONS

In this section, it is presumed that X represents a NAN space, while X' denotes a non-Archimedean Banach space.

Theorem 3.1 Let $g: X \to X'$ be a function that fulfills the inequality (1) and $g(0)=0$.

Let $\xi: X^2 \to [0,\infty)$ be a mapping such that

$$\lim_{n\to\infty} \frac{\xi(4^n x, 4^n y)}{|16|^n} = 0, \forall\, x,y \in X. \tag{3}$$

Then, there is only one quadratic function $Q_2(x)$ from X to X' exists and

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}} \left\{ \frac{1}{|16|^{n+1}} \xi(4^n x, 4^n x) \right\}, \forall\, x \in X. \tag{4}$$

Proof:

Substitute y by x in (1), we arrive

$$\begin{gathered} \|g(2x)+g(4x)-10g(x)-10g(x)\| \le \xi(x,x) \\ \|g(4x)-16g(x)\| \le \xi(x,x), \forall x \in X. \\ \left\| \frac{g(4x)}{16} - g(x) \right\| \le \frac{1}{|16|} \xi(x,x) \end{gathered} \tag{5}$$

Replacing x by $4^n x$ in (5), we get

$$\begin{gathered} \|g(4.4^n x) - 16 g(4^n x)\| \le \xi(4^n x, 4^n x) \\ \left\| \frac{g(4^{n+1}x)}{16^{n+1}} - \frac{g(4^n x)}{16^n} \right\| \le \frac{\xi(4^n x, 4^n y)}{|16|^{n+1}} \end{gathered} \tag{6}$$

Hence the sequence $\left\{ \frac{g(4^n x)}{16^n} \right\}$ is Cauchy.

Therefore $\left\{\frac{g(4^n x)}{16^n}\right\}$ is convergent.
Let

$$Q_2(x) = \lim_{n\to\infty} \frac{g(4^n x)}{16^n}.$$

Consider,

$$\begin{aligned}\left\|\frac{g(4^n x)}{16^n} - g(x)\right\| &= \left\|\frac{g(4^n x)}{16^n} - \frac{g(4^{n-1}x)}{16^{n-1}} + \cdots + \frac{g(4x)}{16} - g(x)\right\| \\ &\le \max\left\{\left\|\frac{g(4^n x)}{16^n} - \frac{g(4^{n-1}x)}{16^{n-1}}\right\|, \cdots, \left\|\frac{g(4x)}{16} - g(x)\right\|\right\} \\ &\le \max\left\{\frac{\xi(4^{n-1}x, 4^{n-1}x)}{|16|^n}, \cdots, \frac{\xi(x,x)}{|16|}\right\} \\ &\le \sup_{0\le m<n}\left\{\frac{\xi(4^m x, 4^m x)}{|16|^{m+1}}\right\} \quad \forall\, n\in\mathbb{N}\ and\ all\ x\in X.\end{aligned} \tag{7}$$

Taking limit $n\to\infty$ on bothsides, we get

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}}\left\{\frac{1}{|16|^{n+1}}\xi(4^n x, 4^n x)\right\}, \forall\, x\in X.$$

Now, by putting x and y as $4^n x$ and $4^n y$ respectively, in (1), we get

$$\|g(3.4^n x - 4^n y) + g(4^n x + 3.4^n y) - 10g(4^n x) - 10g(4^n y)\| \le \xi(4^n x, 4^n y)$$

$$\left\|g\big(4^n(3x-y)\big) + g\big(4^n(x+3y)\big) - 10g(4^n x) - 10g(4^n y)\right\| \le \xi(4^n x, 4^n y)$$

$$\left\|\frac{g\big(4^n(3x-y)\big)}{16^n} + \frac{g\big(4^n(x+3y)\big)}{16^n} - \frac{10g(4^n x)}{16^n} - \frac{10g(4^n y)}{16^n}\right\| \le \frac{\xi(4^n x, 4^n y)}{|16|^n}, \forall x,y\in X.$$

Taking limit $n\to\infty$ on bothsides,

$$\left\|Q_2(3x-y) + Q_2(x+3y) - 10Q_2(x) - 10Q_2(y)\right\| \le \lim_{n\to\infty}\frac{\xi(4^n x, 4^n y)}{|16|^n} = 0.$$

Therefore Q_2 is quadratic mapping.

To prove uniqueness:

$$\begin{aligned}&\| Q_2(x) - Q_2'(x) \| \\ &\le \max\frac{1}{|16|^n}\{\|Q_2(4^n x) - g(4^n x)\|, \|g(4^n x) - Q_2'(4^n x)\|\} \\ &\le \sup_{j\in\mathbb{N}}\left\{\frac{1}{|16|^{n+j+1}}\xi(4^{n+j}x, 4^{n+j}x)\right\}.\end{aligned}$$

Taking limit $n\to\infty$ on bothsides, $Q_2(x) = Q_2'(x)$.
Hence, uniqueness of Q_2 is proved.

Theorem 3.2 Let $g: X\to X'$ be a function that fulfills the inequality (1) and $g(0)=0$.
Let $\xi: X^2\to[0,\infty)$ be a mapping such that

$$\lim_{n\to\infty}|16|^n \xi\left(\frac{x}{4^n}, \frac{y}{4^n}\right) = 0, \forall x,y\in X. \tag{8}$$

Then, there is only one quadratic function $Q_2(x)$ from X to X' exists and

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}}\left\{|16|^{n+1}\xi\left(\frac{x}{4^n}, \frac{x}{4^n}\right)\right\}, \forall x\in X. \tag{9}$$

Proof:
Substitute y by x in (1), we get

$$\begin{aligned}\|g(2x) + g(4x) - 10g(x) - 10g(x)\| &\le \xi(x,x) \\ \|g(4x) - 16g(x)\| &\le \xi(x,x), \forall x\in X.\end{aligned} \tag{10}$$

Replacing x by $\frac{x}{4^{n+1}}$ in (10), we get

$$\begin{aligned}\left\|g\left(4\ \frac{x}{4^{n+1}}\right) - 16g\left(\frac{x}{4^{n+1}}\right)\right\| &\le \xi\left(\frac{x}{4^{n+1}}, \frac{x}{4^{n+1}}\right) \\ \left\|g\left(\frac{x}{4^n}\right) - 16g\left(\frac{x}{4^{n+1}}\right)\right\| &\le \xi\left(\frac{x}{4^{n+1}}, \frac{x}{4^{n+1}}\right) \\ \left\|16^{n+1}g\left(\frac{x}{4^{n+1}}\right) - 16^n g\left(\frac{x}{4^n}\right)\right\| &\le |16|^n \xi\left(\frac{x}{4^{n+1}}, \frac{x}{4^{n+1}}\right)\end{aligned} \tag{11}$$

Hence the sequence $\left\{16^n g\left(\frac{x}{4^n}\right)\right\}$ is Cauchy.
Therefore $\left\{16^n g\left(\frac{x}{4^n}\right)\right\}$ is convergent.
Let

$$Q_2(x) = \lim_{n\to\infty} 16^n g\left(\frac{x}{4^n}\right).$$

Consider,

$$\begin{aligned}&\left\|16^{n+1}g\left(\frac{x}{4^{n+1}}\right) - g(x)\right\| \\ &= \left\|16^{n+1}g\left(\frac{x}{4^{n+1}}\right) - 16^n g\left(\frac{x}{4^n}\right) + 16^n - 16^{n-1}g\left(\frac{x}{4^{n-1}}\right) + 16g\left(\frac{x}{4}\right) - g(x)\right\| \\ &\le \max\left\{\left\|16^{n+1}g\left(\frac{x}{4^{n+1}}\right) - 16^n g\left(\frac{x}{4^n}\right)\right\|, \cdots, \left\|16g\left(\frac{x}{4}\right) - g(x)\right\|\right\} \\ &\le \max\left\{|16|^n\xi\left(\frac{x}{4^{n+1}}, \frac{x}{4^{n+1}}\right), \cdots, |16|^0\xi\left(\frac{x}{4}, \frac{x}{4}\right)\right\} \\ &\le \sup_{0\le m<n}\left\{|16|^{m+1}\xi\left(\frac{x}{4^{m+2}}, \frac{x}{4^{m+2}}\right)\right\} \\ &\qquad \forall\, n\in\mathbb{N}\ \text{and}\ all\ x\in X.\end{aligned} \tag{12}$$

Taking limit $n\to\infty$ on bothsides, we get

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}}\left\{|16|^n\xi\left(\frac{x}{4^{n+1}}, \frac{x}{4^{n+1}}\right)\right\}, \forall\, x\in X.$$

Now, by putting x and y as $\frac{x}{4^{n+1}}$ and $\frac{y}{4^{n+1}}$ respectively, in (1), we get

$$\left\|g\left(3.\frac{x}{4^{n+1}} - \frac{y}{4^{n+1}}\right) + g\left(\frac{x}{4^{n+1}} + 3.\frac{y}{4^{n+1}}\right) - 10g\left(\frac{x}{4^{n+1}}\right) - 10g\left(\frac{y}{4^{n+1}}\right)\right\| \le \xi\left(\frac{x}{4^{n+1}}, \frac{y}{4^{n+1}}\right)$$

$$\left\|g\left(\frac{1}{4^{n+1}}(3x-y)\right) + g\left(\frac{1}{4^{n+1}}(x+3y)\right) - 10g\left(\frac{x}{4^{n+1}}\right) - 10g\left(\frac{y}{4^{n+1}}\right)\right\| \le \xi\left(\frac{x}{4^{n+1}}, \frac{y}{4^{n+1}}\right)$$

$$\left\|16^{\mathrm{n}}.\,g\left(\frac{1}{4^{n+1}}(3x-y)\right) + 16^{\mathrm{n}}.\,g\left(\frac{1}{4^{n+1}}(x+3y)\right) - 16^{\mathrm{n}}.\,10g\left(\frac{x}{4^{n+1}}\right) - 16^{\mathrm{n}}.\,10g\left(\frac{y}{4^{n+1}}\right)\right\| \le |16|^n\xi\left(\frac{x}{4^{n+1}}, \frac{y}{4^{n+1}}\right)$$

Taking limit $n\to\infty$ on both sides,

$$\left\|Q_2(3x-y)+Q_2(x+3y)-10Q_2(x)-10Q_2(y)\right\| \le |16|^n \xi\left(\frac{x}{4^{n+1}},\frac{y}{4^{n+1}}\right)=0.$$

Therefore Q_2 is quadratic mapping.

To prove uniqueness:

$$\| Q_2(x)-Q_2'(x) \| \le \max|16|^n\left\{\left\|Q_2\left(\frac{x}{4^n}\right)-g\left(\frac{x}{4^n}\right)\right\|,\left\|g\left(\frac{x}{4^n}\right)-Q_2'\left(\frac{x}{4^n}\right)\right\|\right\} \le \sup_{j\in\mathbb{N}}\left\{|16|^{n+j+1}\xi\left(\frac{x}{4^{n+j}},\frac{x}{4^{n+j}}\right)\right\}.$$

Taking limit $n\to\infty$ on bothsides, $Q_2(x)=Q_2'(x)$.
Hence, uniqueness of Q_2 is proved.

Corollary 3.3
Let $g: X\to X'$ be a quadratic function that fulfills the inequality,

$$\| Eg(x,y) \| \le \delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s) \forall\, x,y\in X,$$

where $r,s,\delta\in\mathbb{R}^+$
Then,
(i) For $r+s>2$, there is only one quadratic function $Q_2(x): X\to X'$ such that

$$\| g(x)-Q_2(x) \| \le \frac{3\delta}{|16|}\|x\|^{r+s}$$

(ii) For $r+s<2$, there is only one quadratic function $Q_2(x): X\to X'$ such that

$$\| g(x)-Q_2(x) \| \le |16|\, 3\delta\, \|x\|^{r+s}$$

Proof:
Consider,

$$\| Eg(x,y) \| \le \delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s)\ \forall\, x,y\in X.$$

Given

$$\xi(x,y)=\delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s) \tag{13}$$

Case (i) Replacing x and y by $4^n x$ and $4^n x$ respectively in (8), we get

$$\xi(4^n x,4^n x)=\delta\,(\|4^n x\|^{r+s}+\|4^n x\|^{r+s}+\|4^n x\|^r\|4^n x\|^s)=3\delta\|4^n x\|^{r+s} \tag{14}$$

From Theorem (3.1),

$$\|g(x)-Q_2(x)\|\le\sup_{n\in\mathbb{N}}\left\{\frac{1}{|16|^{n+1}}\,\xi(4^n x,4^n x)\right\}\ \forall x\in X.$$
$$\le\sup_{n\in\mathbb{N}}\left\{\frac{1}{|16|^{n+1}}\,3\delta\|4^n x\|^{r+s}\right\}\ \{\text{using (14)}\}$$
$$=3\delta\frac{\|x\|^{r+s}}{|4|^2}\sup_{n\in\mathbb{N}}\left\{|4|^{n[(r+s)-2]}\right\}$$

If $r+s>2$, then we obtain,

$$\|g(x)-Q_2(x)\|\le\frac{3\delta}{|16|}\,\left||x|\right|^{r+s}.$$

Case (ii) Replacing x and y by $\frac{x}{4^n}$ and $\frac{x}{4^n}$ respectively in (13), we get

$$\xi\left(\frac{x}{4^n},\frac{x}{4^n}\right)=\delta\left(\left\|\frac{x}{4^n}\right\|^{r+s}+\left\|\frac{x}{4^n}\right\|^{r+s}+\left\|\frac{x}{4^n}\right\|^{r}\left\|\frac{x}{4^n}\right\|^{s}\right)=3\delta\left\|\frac{x}{4^n}\right\|^{r+s}$$

$$\|g(x)-Q_2(x)\|\le\sup_{n\in\mathbb{N}}\left\{|16|^{n+1}\,\xi\left(\frac{x}{4^n},\frac{x}{4^n}\right)\right\}\ \forall x\in X.$$
$$\le\sup_{n\in\mathbb{N}}\left\{|16|^{n+1}\,3\delta\left\|\frac{x}{4^n}\right\|^{r+s}\right\}$$
$$=3\delta\,\|x\|^{r+s}\sup_{n\in\mathbb{N}}\left\{|4|^2\,|4|^{n[2-(r+s)]}\right\}$$

If $r+s<2$, then we obtain,

$$\|g(x)-Q_2(x)\|\le\ 3\delta\,|16|\,\left||x|\right|^{r+s}.$$

In the situation where $r+s=2$, the following counterexample is applicable.

Example 3.4 Consider a prime number $p>2$ and $g:\mathbb{Q}_p\to\mathbb{Q}_p$ be defined by $g(x)=x^2+1$. For all $n\in\mathbb{N}$, $|2^n|_p=1$, for $\delta>0$, we obtain

$$\|E\,g(x,y)\|=|20|\le 1\le\delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s)\ \forall\, x,y\in X.$$

and

$$\left\|\frac{g(4^{n+1}x)}{16^{n+1}}-\frac{g(4^n x)}{16^n}\right\|=|15|\neq 0$$

which implies $\left\{\frac{g(4^n x)}{16^n}\right\}$ is not a Cauchy sequence.

4. STABILITY OF THE GENERAL Q.F EQUATION

Theorem 4.1 Let g be a function X to X' that satisfies (2) and $g(0)=0$. Let $\xi: X^2\to[0,\infty)$ be a mapping such that

$$\lim_{n\to\infty}\frac{\xi(u^n x,\,u^n y)}{|u^2|^n}=0,\forall x,y\in X, where\ u=v+1. \tag{15}$$

Then, there is only one quadratic function Q_2 from X to X' exists and

$$\|g(x)-Q_2(x)\|\le\sup_{n\in\mathbb{N}}\left\{\frac{1}{|u^2|^{n+1}}\xi(u^n x,u^n x)\right\},\ \forall\, x\in X. \tag{16}$$

Proof:
Putting $y=x$ in (2), we arrive

$$\|g(ux)-u^2 g(x)\|\le\xi(x,x) \tag{17}$$

Substituting x by $u^n x$ in (4.3), we get

$$\|g(u^{n+1}x)-u^2 g(u^n x)\|\le\xi(u^n x,u^n x)$$

$$\left\|\frac{g(u^{n+1}x)}{u^{2n+2}}-\frac{g(u^n x)}{u^{2n}}\right\|\le\frac{\xi(u^n x,\,u^n x)}{|u|^{2n+2}} \tag{18}$$

Hence the sequence $\left\{\frac{g(u^n x)}{u^{2n}}\right\}$ is Cauchy.

Therefore $\left\{\frac{g(u^n x)}{u^{2n}}\right\}$ is convergent.
Let

$$Q_2(x) = \lim_{n\to\infty} \frac{g(u^n x)}{u^{2n}}.$$

Consider,

$$\begin{aligned}
&\left\|\frac{g(u^n x)}{u^{2n}} - g(x)\right\| \\
&\le \max\left\{\left\|\frac{g(u^n x)}{u^{2n}} - \frac{g(u^{n-1}x)}{u^{2n-2}}\right\|, \cdots, \left\|\frac{g(ux)}{u^2} - g(x)\right\|\right\} \\
&\le \max\left\{\frac{\xi(u^{n-1}x, u^{n-1}x)}{|u|^{2n}}, \cdots, \frac{\xi(x,x)}{|u|^2}\right\} \\
&\le \sup_{0\le m<n} \left\{\frac{\xi(u^m x, u^m x)}{|u^2|^{m+1}}\right\} \forall n \in \mathbb{N} \text{ and } x \in X.
\end{aligned} \tag{19}$$

Taking limit $n \to \infty$ on bothsides, we get

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}} \left\{\frac{1}{|u^2|^{n+1}} \xi(u^n x, u^n x)\right\}, \forall x \in X.$$

Now, by substituting x by $u^n x$ and y by $u^n y$ respectively, in (2), we get

$$\begin{aligned}
&\|g(u-1)u^n x - u^n y) + g(u^n x + (u-1)u^n y) \\
&-((u-1)^2+1)g(u^n x) - ((u-1)^2+1)g(u^n y)\| \\
&\le \xi(u^n x, u^n y)
\end{aligned}$$

$$\begin{aligned}
&\| g((u-1)^2+1)((u-1)x - y) + g((u-1)^2+1) \\
&(x + (u-1)y) - ((u-1)^2+1)g((u-1)^2+1)x \\
&-((u-1)^2+1)g((u-1)^2+1)y \| \le \xi(u^n x, u^n y)
\end{aligned}$$

$$\left\|\frac{g(u^n(u-1)x-y)}{u^{2n}} + \frac{g\left(u^n(x+(u-1)y)\right)}{u^{2n}} - \frac{((u-1)^2+1)g(u^n x)}{u^{2n}} - \frac{((u-1)^2+1)g(u^n y)}{u^{2n}}\right\| \le \frac{\xi(u^n x, u^n y)}{|u^2|^n}$$

Taking limit $n \to \infty$ on bothsides,

$$\|Q_2(u-1)x - y) + Q_2(x + (u-1)y) - ((u-1)^2+1)\, Q_2(x) - ((u-1)^2+1)Q_2(y)\| \le \lim_{n\to\infty} \frac{\xi(u^n x, u^n y)}{|u^2|^n} = 0.$$

Therefore Q_2 is quadratic mapping.

To prove uniqueness:

$$\begin{aligned}
&\| Q_2(x) - Q_2'(x) \| \le \max \frac{1}{|u^2|^n} \{\| Q_2(u^n x) - g(u^n x) \|, \\
&\qquad \| g(u^n x) - Q_2'(u^n x) \|\} \\
&\| Q_2(x) - Q_2'(x) \| \le \sup_{j\in\mathbb{N}} \left\{\frac{1}{|u^2|^{n+j+1}} \xi(u^{n+j}x, u^{n+j}x)\right\}.
\end{aligned}$$

Taking limit $n \to \infty$ on bothsides, $Q_2(x) = Q_2'(x)$.
Hence, uniqueness of Q_2 is proved.

Theorem 4.2 Let g be a function X to X' that satisfies (2) and $g(0) = 0$. Let $\xi: X^2 \to [0, \infty)$ be a mapping such that

$$\lim_{n\to\infty} |u|^{2n} \xi\left(\frac{x}{u^n}, \frac{y}{u^n}\right) = 0, \forall x, y \in X \tag{20}$$

where, u=υ+1, then, there is only one quadratic function Q_2 from X to X' exists and

$$\| g(x) - Q_2(x) \| \le \sup_{n\in\mathbb{N}} \left\{|u^2|^{n+1} \xi\left(\frac{x}{u^n}, \frac{x}{u^n}\right)\right\}, \forall x, y \in X. \tag{21}$$

Proof:
Putting $y = x$ in (2), we arrive

$$\|g(ux) - u^2 g(x)\| \le \xi(x, x) \tag{22}$$

Substituting x by $\frac{x}{u^{n+1}}$ in (22), we get

$$\begin{aligned}
&\left\|g\left(u\frac{x}{u^{n+1}}\right) - u^2 g\left(\frac{x}{u^{n+1}}\right)\right\| \le \xi\left(\frac{x}{u^{n+1}}, \frac{x}{u^{n+1}}\right) \\
&\left\|u^{2(n+1)} g\left(\frac{x}{u^{n+1}}\right) - u^{2n} g\left(\frac{x}{u^n}\right)\right\| \le |u|^{2n} \xi\left(\frac{x}{u^{n+1}}, \frac{x}{u^{n+1}}\right)
\end{aligned} \tag{23}$$

Hence the sequence $\left\{|u|^{2n} g\left(\frac{x}{u^n}\right)\right\}$ is Cauchy.
Therefore $\left\{|u|^{2n} g\left(\frac{x}{u^n}\right)\right\}$ is convergent.
Let

$$Q_2(x) = \lim_{n\to\infty} |u|^{2n} g\left(\frac{x}{u^n}\right)$$

Consider

$$\begin{aligned}
&\left\|u^{2(n+1)}\, g\left(\frac{x}{u^{n+1}}\right) - g(x)\right\| \\
&\le \left\{\left\|u^{2(n+1)}\, g\left(\frac{x}{u^{n+1}}\right) - u^{2n} g\left(\frac{x}{u^n}\right)\right\|, \ldots \left\|u^{2n} g\left(\frac{x}{u^n}\right) - g(x)\right\|\right\} \\
&\le max \left\{|u|^{2n} \xi\left(\frac{x}{u^{n+1}}, \frac{x}{u^{n+1}}\right), \ldots |u|^{2n} \xi\left(\frac{x}{u}, \frac{x}{u}\right)\right\} \\
&\le \sup_{0\le m<n} \left\{|u|^{2(m+1)} \xi\left(\frac{x}{u^{m+2}}, \frac{x}{u^{m+2}}\right)\right\} \\
&\qquad \forall n \in \mathbb{N} \text{ and } x \in X.
\end{aligned} \tag{24}$$

Now, by substituting x by $\frac{x}{u^{n+1}}$and y by $u^n y$ respectively, in (2), we get

$$\|g\left(\upsilon.\frac{x}{u^{n+1}} - \frac{y}{u^{n+1}}\right) + g\left(\frac{x}{u^{n+1}} + \upsilon.\frac{y}{u^{n+1}}\right) - (\upsilon^2 + 1)g\left(\frac{x}{u^{n+1}}\right) - (\upsilon^2+1)g\left(\frac{y}{u^{n+1}}\right) \| \le \xi\left(\frac{x}{u^{n+1}}, \frac{y}{u^{n+1}}\right)$$

$$\|g\left(\frac{1}{u^{n+1}}(\upsilon x - y)\right) + g\left(\frac{1}{u^{n+1}}(\upsilon x + y)\right) - (\upsilon^2 + 1)g\left(\frac{x}{u^{n+1}}\right) - (\upsilon^2+1)g\left(\frac{y}{u^{n+1}}\right) \| \le \xi\left(\frac{x}{u^{n+1}}, \frac{y}{u^{n+1}}\right)$$

$$\left\|u^{2n}\, g\left(\frac{1}{u^{n+1}}(\upsilon x - y)\right) + u^{2n} g\left(\frac{1}{u^{n+1}}(\upsilon x + y)\right) - (\upsilon^2+1)u^{2n} g\left(\frac{x}{u^{n+1}}\right) - (\upsilon^2+1)u^{2n} g\left(\frac{y}{u^{n+1}}\right)\right\| \le u^{2n} \xi\left(\frac{x}{u^{n+1}}, \frac{y}{u^{n+1}}\right)$$

$$\|Q_2(\upsilon x - y) + Q_2(x + \upsilon y) - (\upsilon^2+1)Q_2(x) - (\upsilon^2+1)Q_2(y)\| \le \lim_{n\to\infty} |u|^{2n} \xi\left(\frac{x}{u^{n+1}}, \frac{y}{u^{n+1}}\right) = 0.$$

Therefore Q_2 is quadratic mapping.

To prove uniqueness:

$$\begin{aligned}
&\| Q_2(x) - Q_2'(x)\| \\
&\le \max |u|^{2n} \left\{\| Q_2\left(\frac{x}{u^n}\right) - g\left(\frac{x}{u^n}\right) \|, \|g\left(\frac{x}{u^n}\right) - Q_2'\left(\frac{x}{u^n}\right) \|\right\} \\
&\le \sup_{j\in\mathbb{N}} \left\{|u|^{2(n+j+1)} \xi\left(\frac{x}{u^{n+j}}, \frac{x}{u^{n+j}}\right)\right\}.
\end{aligned}$$

Taking limit $n \to \infty$ on bothsides, $Q_2(x) = Q_2'(x)$.

Hence, uniqueness of Q_2 is proved.

Corollary 4.3

Let $g: X \to X'$ be a quadratic function that fulfills the inequality,

$$\| Dg(x,y) \| \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r \|y\|^s) \forall\, x, y \in X,$$

where $r, s, \delta \in \mathbb{R}^+$

Then,

(i) For $r+s>2$, there is only one quadratic function $Q_2(x): X \to X'$ such that

$$\| g(x) - Q_2(x) \| \leq \frac{3\delta}{|u|^2} \|x\|^{r+s}$$

(ii) For $r+s<2$, there is only one quadratic function $Q_2(x): X \to X'$ such that

$$\| g(x) - Q_2(x) \| \leq |u|^2\, 3\delta\, \|x\|^{r+s}.$$

Proof:

Consider,

$$\| Eg(x,y) \| \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r \|y\|^s) \quad \forall\, x, y \in X. \tag{25}$$

Given $\xi(x,y) = \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r \|y\|^s)$

Case (i) Replacing x and y by $u^n x$ and $u^n x$ respectively in (20), we get

$$\xi(u^n x, u^n x) = \delta(\|u^n x\|^{r+s} + \|u^n x\|^{r+s} + \|u^n x\|^r \|u^n x\|^s) = 3\delta \|u^n x\|^{r+s} \tag{26}$$

From Theorem 4.1,

$$\begin{aligned}\|g(x) - Q_2(x)\| &\leq \sup_{n\in\mathbb{N}} \left\{\frac{1}{|u^2|^{n+1}}\, \xi(u^n x, u^n x)\right\} \forall\, x \in X.\\ &\leq \sup_{n\in\mathbb{N}} \left\{\frac{1}{|u^2|^{n+1}}\, 3\delta \|u^n x\|^{r+s}\right\} \text{using (4.12)}\}\\ &= 3\delta \frac{\|x\|^{r+s}}{|u|^2} \sup_{n\in\mathbb{N}} \left\{|u|^{n[(r+s)-2]}\right\}\end{aligned}$$

If $r+s>2$, then we obtain, $\|g(x) - Q_2(x)\| \leq 3\delta \frac{\|x\|^{r+s}}{|u|^2}$

Case (ii) Replacing x and y by $\frac{x}{u^n}$ and $\frac{x}{u^n}$ respectively in (25), we get

$$\begin{aligned}\xi\left(\frac{x}{u^n}, \frac{x}{u^n}\right) &= \delta\left(\left\|\frac{x}{u^n}\right\|^{r+s} + \left\|\frac{x}{u^n}\right\|^{r+s} + \left\|\frac{x}{u^n}\right\|^r \left\|\frac{x}{u^n}\right\|^s\right)\\ &= 3\delta \left\|\frac{x}{u^n}\right\|^{r+s}\end{aligned}$$

$$\begin{aligned}\|g(x) - Q_2(x)\| &\leq \sup_{n\in\mathbb{N}} \left\{|u^2|^{n+1}\, \xi\left(\frac{x}{u^n}, \frac{x}{u^n}\right)\right\} \forall\, x \in X.\\ &\leq \sup_{n\in\mathbb{N}} \left\{|u^2|^{n+1}\, 3\delta \left\|\frac{x}{u^n}\right\|^{r+s}\right\}\\ &= 3\delta\, \|x\|^{r+s} \sup_{n\in\mathbb{N}} \left\{|u|^{n[2-(r+s)]}\right\}\end{aligned}$$

If $r+s<2$, then we obtain,

$$\|g(x) - Q_2(x)\| \leq 3\delta\, |u|^2\, ||x||^{r+s}.$$

For the scenario where $r+s=2$, we present the following counterexample.

Example 4.4 Consider a prime number $p>2$ and $g: \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $g(x) = x^2 + 1$. For all $n \in \mathbb{N}$, $|2^n|_p = 1$, for $\delta > 0$, we obtain

$$\|Dg(x,y)\| = |v|^2 \leq 1 \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r \|y\|^s) \quad \forall\, x, y \in X.$$

and

$$\left\|\frac{g(u^{n+1}x)}{u^{2n+2}} - \frac{g(u^n x)}{u^{2n}}\right\| = \frac{|1-u^2|}{|u|^{2n+2}} \neq 0,$$

where $u = v+1$, which implies $\left\{\frac{g(u^n x)}{u^{2n}}\right\}$ is not a Cauchy sequence.

5. CONCLUSION

In recent years, numerous authors have deliberated on the HUS (Hyers-Ulam stability) of functional equations in NAN space. In the present article, we delve into the HUS of the generalized Q.F. equation

$$g(vx-y) + g(x+vy) - (v^2+1)g(x) - (v^2+1)g(y)$$

in NAN space. Also, we proved the theorem using direct method. This method plays a major role in proving the stability problems. Also, we have proved the result for particular case with some suitable counter examples. This work contributes to the broader understanding of stability problems in the context of NAN spaces and functional equations.

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