International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 11, No. 2, February, 2024, pp. 446-456
Journal homepage: http://iieta.org/journals/mmep

Optimizing Two-Warehouse Inventory for Shelf-Life Stock with Time-Varying Bi-Quadratic Demand Under Shortages and Inflation

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https://doi.org/10.18280/mmep.110216

Received: 14 July 2023
Revised: 13 October 2023
Accepted: 22 October 2023
Available online: 27 February 2024

Keywords:
shelf-life stock, deterioration, inflation, biquadratic time-dependent demand, partial backlogging

ABSTRACT

An organisation needs to control its inventory efficiently and goods shelf life is a key factor. Goods can deteriorate due to various factors such as damage, rotting, and dryness, reducing their utility over time. The shelf life of goods refers to the maximum period for which they can be stored while maintaining their acceptable quality. This research paper focuses on selecting the best replenishment strategy for shelf-life stock with biquadratic time-dependent demand, accounting for inflation and shortages, where shortages are partially backlogged. The objective is to minimize the overall cost, which includes several inventory costs, using MATLAB to optimize the quantity and time. The investigation indicates that the average total cost is $483, the optimal order quantity is 461 units, and the replenishment quantity is 538 units, which occurs at a replenishment interval of 3.7 years. The model's elucidation is enhanced by a numerical example and a sensitivity analysis.

1. INTRODUCTION

Inventory management represents a cornerstone of operational excellence within the realm of business. Its fundamental role encompasses the intricate orchestration of goods, encompassing their production, sale, or procurement, all designed to ensure the seamless functioning of an organization. Complementing this indispensable operational facet is the power of mathematical models, a systematic tool enabling the elucidation, prediction, and control of a spectrum of internal and external processes. Across various industries, from management to operations research and engineering, these models find their application, shaping strategies and guiding decisions.

Central to the domain of inventory management stands the concept of Economic Order Quantity (EOQ), a methodology wielding the capacity to ascertain the optimal quantity of goods to be produced or procured, predicated upon the mean rate of inventory consumption. This precision-driven approach, rooted in demand analysis and foresight of usage patterns, carries the dual objective of cost containment and the augmentation of customer satisfaction.

The resonance of effective inventory administration reverberates deeply throughout the expanse of organizational performance and profitability. This resonance resonates even more profoundly within industries dealing with perishable goods, where the slightest degradation in product quality can engender ripples of discontent among consumers, tumultuous shifts in sales, and unwarranted escalations in operational overheads. This degradation assumes diverse forms, from deterioration to spoilage, moisture loss to vaporization, all conspiring to diminish product quality and, subsequently, customer contentment. In response, astute adjustments in inventory management strategies, ordering protocols, and replenishment schemes become imperatives.

Yet, within the tapestry of inventory management, another thread of consideration, equally essential, is the specter of inflation. Inflation, the inexorable rise in the cost of goods and services over time, inexorably erodes the purchasing prowess of currency. Its impact on inventory control unfolds along multifaceted trajectories. Firstly, the escalating costs entailed in producing, transporting, and warehousing perishable commodities can exert monumental pressures on inventory maintenance expenses. Heightened prices of raw materials and energy sources may necessitate consumer price increments. Simultaneously, surging fuel expenses and ballooning rents contribute to the overall overhead of storing and distributing perishable products. The consequence: amplified inventory holding costs, exerting perturbations on the financial bottom line.

Secondly, inflation wields the power to influence shifts in the demand for perishable goods. In a climate of inflationary growth coupled with diminished real wages, the demand for such goods may surge, mandating elevated inventory levels and more frequent restocking. Striking the balance between maintaining a sufficient stock to cater to fluctuating demand, while averting excess inventory that translates into augmented holding costs and wastage, metamorphoses into a formidable endeavor.

Conversely, subdued inflation may engender a wane in the appetite for perishable products, leading to lower inventory levels and diminished aggregate order volume. Businesses previously equipped to meet high demand levels may grapple with underutilized infrastructure, a factor that can impact profitability as fixed costs persist.

In response to these market dynamics, the proposed model is committed to improving inventory management for shelf-life goods, accounting for factors including inflation, partial backlogging, and bi-quadratic demand. As our primary objective, we aim to achieve cost minimization while optimizing both quantity and time, providing a distinctive perspective in the domain of shelf-life inventory management.

1.1 Role of MATLAB in constrained non-linear minimization

MATLAB, an abbreviation for MATrix LABoratory, stands as an exceptionally versatile and widely-utilized software platform that has earned a reputation for its exceptional capabilities in numerical computing, data analysis, algorithm development, and data visualization. It emerges as an indispensable tool, revered by researchers and professionals hailing from a broad spectrum of disciplines, encompassing mathematics, engineering, physics, and numerous others. This comprehensive software platform presents a formidable array of resources, adeptly addressing the intricate computational challenges encountered within these diverse fields.

Within the context of our research paper, MATLAB assumes a central and pivotal role, particularly when tasked with addressing intricate constrained non-linear minimization problems. To achieve this, we rely on the optimization toolbox integrated into MATLAB, which offers a set of robust tools explicitly designed for handling such complex tasks. In particular, our research leverages the capabilities of the "fmincon" solver, known for its high efficiency in managing constrained nonlinear minimization problems.

2. LITERATURE REVIEW

A successful business must effectively manage its inventory, and this is especially crucial when it comes to goods that are deteriorating. But managing deteriorating inventory can be difficult since there are so many things to take into account, including shifts in demand, holding costs, backlogs, shortages, and inflation. Numerous researchers have created models that attempt to optimise inventory management for degrading items by taking into consideration these issues in response to these difficulties.

Mishra and Singh [1] employed a computational approach to optimize the total cost function of an inventory model that accounts for ramp-type demand and linear deterioration. Venkateswarlu and Mohan [2] developed a deterministic inventory model for deteriorating items that integrates quadratic demand functions and proportional deterioration rates. Meanwhile, Chauhan and Singh [3] investigated optimal replenishment and ordering policies for time-varying deterioration items with varying demand, utilizing a discounted cash flow approach.

A model for calculating the Economic Order Quantity (EOQ) for non-instantaneously deteriorating items with stock-dependent demand, inflation, and partial backlogging was introduced by Palanivel et al. [4], which utilizes two warehouses. In a separate study, Kumar and Chanda [5] presented a two-warehouse inventory model specifically designed for technology products. Yadav and Swami [6] developed a model for non-instantaneous deteriorating items that considers rented and owned warehouses.

Shaikh et al. [7] suggested a two-warehouse inventory model that considers partial backlogging and advanced payment. Taghizadeh-Yazdi et al. [8] proposed a mathematical programming model that maximizes the profit of suppliers, manufacturers, and distributors in a three-echelon supply chain, accounting for the deteriorating nature of raw materials and final products. Meanwhile, Khan et al. [9] presented a two-storage inventory model with advance payment, where demand is dependent on the selling price. The model also takes into account partial shortages with a fixed backlogging rate.

Mashud [10] proposed an Economic Order Quantity (EOQ) inventory model that considers deteriorating items with stock-dependent demand and full backlogged shortages, while also taking into account price changes. Suman [11] presented a deterministic inventory model for deteriorating items with a biquadratic demand function over time and allowing for shortages. A strategy where suppliers give price discounts to retailers that make advance payments was presented in Duary et al. 's [12] study.

The review focuses on the problems that companies run into while trying to manage deteriorating inventories and the models that have been put out in various studies to solve these problems. By taking into account elements like demand volatility, holding costs, backlogs, shortages, and inflation, these models can aid in the optimisation of inventory decisions. These research' conclusions offer useful information about how to manage inventory for degrading goods, which can aid firms in making wise choices. In light of the effects of price variations on overall profit, recent researches have proposed novel models to manage supply chains with uncertain demand and inflation. Businesses can benefit from extra advice from the three-level supply chain model by Padiyar et al. [13] and the Stackelberg game-based model by Mahdavisharif et al. [14] to increase their inventory management and general profitability. The literature review provided corresponds to the data presented in Table 1.

While some prior research has considered shelf-life goods in inventory management models, there remains a research gap in the development of comprehensive inventory models tailored specifically to address the intricate challenges posed by these goods. The existing literature may have touched on aspects of shelf-life management, but opportunities exist to refine and expand these models further. This research aims to bridge this gap by presenting an advanced and holistic inventory model that comprehensively accounts for factors specific to goods with a limited shelf life, such as biquadratic time-dependent demand, inflation, and partial backlogged shortages.

The motivation behind this research is deeply rooted in the pressing need for efficient inventory management tailored specifically to goods characterized by a limited shelf life. Effectively managing these products poses a complex challenge, one that involves the careful consideration of several crucial cost components, namely holding cost, deterioration cost, shortage cost, and lost sale cost.

At its core, the primary aim of this research is to craft a comprehensive inventory model meticulously designed to optimize the replenishment strategy for shelf-life goods, all while intricately addressing these fundamental cost elements. The ultimate aspiration is to provide invaluable assistance to companies in making judicious inventory decisions, ultimately culminating in the minimization of these indispensable costs.

This research introduces a novel inventory model meticulously crafted to enhance the replenishment plan for products subjected to the constraints of a limited shelf life. It meticulously factors in a spectrum of variables, including the intricacies of biquadratic time-dependent demand, the influence of inflation, and the implications of partial backlogged shortages. The paramount significance of this work lies in its profound ability to amalgamate these critical components into a unified and comprehensive model. In stark contrast to prior studies, which often scrutinized these factors in isolation, this research takes on a holistic perspective.

One of the most compelling advantages inherent in the adoption of this model lies in its transformative capacity to guide companies in making well-considered inventory decisions, subsequently leading to the minimization of operational costs and the overarching enhancement of organizational efficiency.

Table 1. Literature review for the proposed model

Authors / Year	Warehouse System	Inflation	Shortage	Demand	Deterioration
Mishra and Singh (2012)	Single	No	Not allowed	Ramp-type	Instantaneous
Venkateswarlu and Mohan (2013)	Single	No	Fully backlogged	Quadratic	Instantaneous
Chauhan and Singh (2014)	Two-warehouse	Yes	Partial Backlogging	Linearly time dependent	Instantaneous
Palanivel et al. (2016)	Two-warehouse	Yes	Partial Backlogging	Stock dependent	Non-instantaneous
Kumar and Chanda (2018)	Two-warehouse	No	Not allowed	Known and governed by innovation process	Instantaneous
Yadav and Swami (2019)	Two-warehouse	No	Fully backlogged	Linearly time dependent	Non-instantaneous
Shaikh (2019)	Two-warehouse	No	Partially backlogging	Price dependent	Instantaneous
Taghizadeh et al. (2020)	Single	No	Partial Backlogging	Price-dependent	Instantaneous
Khan and Shaikh (2020)	Two-warehouse	No	Partial Backlogging	Price-dependent	Non-instantaneous
Mashud (2020)	Single	No	fully backlogged	Multiple	Instantaneous
Suman (2021)	Single	No	fully backlogged	Biquadrate	Instantaneous
Duary et al. (2022)	Two-warehouse	No	Partial Backlogging	Selling price, time and frequency of advertisement dependent	Instantaneous
Mahdavisharif et al. (2022)	Single	No	Partial Backlogging	Price and time	Instantaneous
Padiyar et al. (2022)	Single	Yes	Not allowed	Constant	Instantaneous
In this paper	Two-warehouse	Yes	Partial Backlogging	Biquadratic time-dependent	Non-instantaneous

3. ASSUMPTIONS

The model introduced in this study is formulated based on the following notation and assumptions.

(1). The demand in this study is represented as a biquadratic function of time.

$$\text{i.e. } D(t)=\begin{cases}\alpha+\beta t^4 & ,I(t)\ge 0\\ \alpha & ,I(t)<0\end{cases}$$

where, $\alpha, \beta>0$

(2). Deterioration are not allowed in RW whereas deterioration occurs in OW with constant rate $0<\zeta<1$ at time $t\in[t_2, t_3]$.

(3). Holding cost per unit time and deterioration cost per unit time are constant.

(4). Considering the continuous increase of t_2 compared to t_1 and t_3 compared to t_2, it is reasonable to make the assumption that:

$$t_1=l_1t_2,\ t_2=l_2t_3,\ t_1=l_1l_2t_2$$

where $l_1 l_2\in(0,1)$.

(5). The replenishment rate is considered to be infinite, implying that there is no lead time, and orders are delivered instantaneously.

(6). The framework accommodates shortages occurring within the time span $t\in[t_3,T]$, during which a partial backlogging mechanism is employed at a constant rate Υ. It is imperative to highlight that the fraction of shortages leading to sales loss is precisely quantified as $1\text{-}\Upsilon$.

(7). The impact of inflation on inventory costs is incorporated into the model.

4. MODEL FORMULATION

The objective of this model formulation is to analyze the inventory management system of a warehouse that stores a lot size of Q units. Among the Q units, L units are placed in an owned warehouse, and the remaining Q-L units are placed in a rented warehouse. The rented warehouse depletes due to demand only, as the product has a shelf-life, and becomes zero during the time period [$0\ t_1$]. On the other hand, the owned warehouse remains constant during this period.

During the time interval [$t_1\ t_2$], the owned warehouse starts depleting due to demand only. After this period, during the time interval [$t_2\ t_3$], the owned warehouse becomes zero due to the combined effect of deterioration and demand. Subsequently, during the time period [$t_3\ T$], shortages occur, and the level of negative inventory during this time interval is represented by $I_s(t)$.

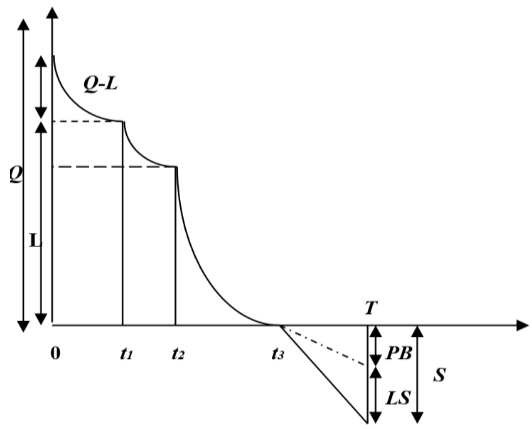

Figure 1. Graphical representation of the proposed two-warehouse inventory model

To better understand the behaviour of this inventory management system over the time interval [$0\ T$], a graphical representation has been provided in Figure 1. This model formulation takes into account various factors such as inventory level, demand, deterioration, and shortage to provide a comprehensive view of the inventory management system of the warehouse.

The governing differential equations for above inventory model are represented by:

$$\frac{dI_r(t)}{dt} = -D(t),\ 0 < t \le t_1 \tag{1}$$

$$\frac{dI_o(t)}{dt} = -D(t),\ \ t_1 \le t \le t_2 \tag{2}$$

$$\frac{dI_o(t)}{dt} + \zeta I_o(t) = -D(t),\ \ t_2 \le t \le t_3 \tag{3}$$

$$\frac{dI_s(t)}{dt} = -D(t)\Upsilon,\ \ t_3 \le t \le T \tag{4}$$

The solutions to the aforementioned differential equation, determined *by* applying the specified boundary conditions $I_r(t_1)=0$, $I_o(t_1)=L$, $I_o(t_3)=0$, $I_s(t_3)=0$ respectively, are as follows:

$$I_r(t) = \frac{\beta t_1^5}{5} + \alpha(t_1 - t) - \frac{\beta t^5}{5},\ \ 0 < t \le t_1 \tag{5}$$

$$I_o(t) = \frac{\beta t_1^5}{5} + \alpha t_1 + L - \alpha t - \frac{\beta t^5}{5},\ \ t_1 \le t \le t_2 \tag{6}$$

$$I_o(t) = \left[\begin{array}{l} \dfrac{e^{-\zeta(t-t_3)}\left(\beta\zeta^4 t_3^4 + \alpha\zeta^4 - 4\beta\zeta^3 t_3^3 + 12\beta\zeta^2 t_2^2 - 24\beta\zeta t_3 + 24\beta\right)}{\zeta^5} \\ -\dfrac{\left(\alpha\zeta^4 + 24\beta\right)}{\zeta^5} - \dfrac{\beta t^4}{\zeta} + \dfrac{4\beta t^3}{\zeta^2} - \dfrac{12\beta t^2}{\zeta^3} + \dfrac{24\beta t}{\zeta^4} \end{array}\right], \tag{7}$$

$$t_2 \le \mathrm{t} \le t_3$$

$$I_s(t) = -\alpha\Upsilon(t - t_3),\ \ t_3 \le t \le T \tag{8}$$

By considering continuity at time $t=t_2$ in the OW, it can be deduced from Eqs. (6) and (7) that:

$$L = \left[\begin{array}{l} \alpha t_2 - \alpha t_1 - \dfrac{\beta t_2^4}{\zeta} + \dfrac{4\beta t_2^3}{\zeta^2} - \dfrac{12\beta t_2^2}{\zeta^3} + \dfrac{24\beta t_2}{\zeta^4} \\ -\dfrac{\alpha\zeta^4 + 24\beta}{\zeta^5} - \dfrac{\beta t_1^5}{5} + \dfrac{\beta t_2^5}{5} + \\ \dfrac{e^{-\zeta(t_2-t_3)}\left(\beta\zeta^4 t_3^4 + \alpha\zeta^4 - 4\beta\zeta^3 t_3^3 + 12\beta\zeta^2 t_3^2 - 24\beta\zeta t_3 + 24\beta\right)}{\zeta^5} \end{array}\right] \tag{9}$$

In the context of RW, there exists a quantity Q-L unit at the initial time t. To calculate the value of Q-L and putting the value of $t=0$ in the (5). we get:

$$I_r(0) = Q - L = \frac{\beta t_1^5}{5} + \alpha t_1$$

Furthermore, by substituting $t=T$ into Eq. (8), we can determine the maximum level of backlogging that occurs per cycle, i.e. $S=-\alpha\Upsilon(T-t_3)$.

Hence, the total quantity to be replenished per cycle can be expressed as follows: $TQC=Q-S$.

$$TQC=\left[\begin{array}{l}\dfrac{e^{-\zeta(t_2-t_3)}\begin{pmatrix}\beta\zeta^4t_3^4+\alpha\zeta^4-4\beta\zeta^3t_3^3\\+12\beta\zeta^2t_3^2-24\beta\zeta t_3+24\beta\end{pmatrix}}{\zeta^5}\\+\dfrac{4\beta t_2^3}{\zeta^2}-\dfrac{12\beta t_2^2}{\zeta^3}+\dfrac{24\beta t_2}{\zeta^4}-\dfrac{\alpha\zeta^4+24\beta}{\zeta^5}\\-\dfrac{\beta t_1^5}{5}+\dfrac{\beta t_2^5}{5}+\dfrac{\beta t_1^5}{5}+\alpha t_1+\alpha\Upsilon(T-t_3)\\-\dfrac{\beta t_2^4}{\zeta}+\alpha t_2-\alpha t_1\end{array}\right] \tag{10}$$

The total cost per cycle comprises the following components:

I. Ordering cost per cycle

$$OC=A$$

II. Inventory holding cost per cycle in the R.W

$$HC_{rw}=H_1\int_0^{t_1}I_r(t)e^{-rt}\,dt$$

$$HC_{rw}=\frac{H_1e^{-rt_1}}{5r^6}\left[\begin{array}{l}120\beta-120\beta e^{rt_1}+5\alpha r^4-5\alpha r^4e^{rt_1}\\+60\beta r^2t_1^2+20\beta r^3t_1^3+5\beta r^4t_1^4+\\120\beta rt_1+\beta r^5t_1^5e^{rt_1}+5\alpha r^5t_1e^{rt_1}\end{array}\right]$$

III. The inventory holding cost per cycle in OW

$$HC_{ow}=H_2\int_0^{t_3}I_o(t)e^{-rt}dt$$

$$HC_{ow}=H_2\left[\int_0^{t_1}I_o(t)e^{-rt}dt+\int_{t_1}^{t_2}I_o(t)e^{-rt}dt+\int_{t_2}^{t_3}I_o(t)e^{-rt}dt\right]$$

$$HC_{ow}=-H_2\left\{\begin{array}{l}\dfrac{\alpha\left(e^{-rt_1}(rt_1+1)-e^{-rt_2}(rt_2+1)\right)}{r^2}+\dfrac{\beta}{5}\left[\begin{array}{l}\dfrac{e^{-rt_1}\left(r^5t_1^5+5r^4t_1^4+20r^3t_1^3+60r^2t_1^2+120rt_1+120\right)}{r^6}\\-\dfrac{e^{-rt_2}\left(r^5t_2^5+5r^4t_2^4+20r^3t_2^3+60r^2t_2^2+120rt_2+120\right)}{r^6}\end{array}\right]\\-\dfrac{L\left(e^{-rt_1}-e^{-rt_2}\right)}{r}-\dfrac{24\beta}{\zeta^4}\left[\dfrac{e^{-rt_2}(rt_2+1)-e^{-rt_3}(rt_3+1)}{r^2}\right]+\dfrac{12\beta}{\zeta^3}\left[\dfrac{\begin{array}{l}e^{-rt_2}\left(r^2t_2^2+2rt_2+2\right)\\-e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)\end{array}}{r^3}\right]\\-\dfrac{4\beta}{\zeta^2}\left[\dfrac{e^{-rt_2}\left(r^3t_2^3+3r^2t_2^2+6rt_2+6\right)-e^{-rt_3}\left(r^3t_3^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]+\\\dfrac{\beta}{\zeta}\left[\dfrac{e^{-rt_2}\left(r^4t_2^4+4r^3t_2^3+12r^2t_2^2+24rt_2+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\-\dfrac{\alpha t_1\left(e^{-rt_1}-e^{-rt_2}\right)}{r}+\dfrac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+r)}+\dfrac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^5(\zeta+r)}+\dfrac{\alpha\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta r}+\\\dfrac{24\beta\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta^5r}-\dfrac{\beta t_1^5\left(e^{-rt_1}-e^{-rt_2}\right)}{5r}+\\\dfrac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+r)}-\dfrac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^2(\zeta+r)}\\+\dfrac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^3(\zeta+r)}-\dfrac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^4(\zeta+r)}+Lt_1\end{array}\right\}$$

IV. Worth shortage cost per cycle

$$SC = sc\int_{t_3}^{T} I_s\left(t\right)e^{-rt}dt$$

$$SC = -sc\left(\frac{\alpha\Upsilon\left(e^{-rT} - e^{-rt_3}\right) + \alpha\Upsilon re^{-rT}\left(T - t_3\right)}{r^2}\right)$$

V. Lost sale cost per cycle under inflation

$$LS = e^{-rT} ls\int_{t_3}^{T} \alpha\left(1-\Upsilon\right)dt$$

$$LS = ls\left(\alpha e^{-rT}\left(T - t_3\right)\left(1-\Upsilon\right)\right)$$

VI. The deterioration cost per cycle in OW

$$DC = D_1\zeta\int_{t_2}^{t_3} I_o(t)e^{-rt}\ dt$$

$$DC = -D_1\zeta\left\{\begin{array}{l}
\frac{12\beta}{\zeta^3}\left[\frac{e^{-rt_2}\left(r^2t_2^2+2rt_2+2\right)-e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)}{r^3}\right]-\frac{24\beta}{\zeta^4}\left[\frac{e^{-rt_2}\left(rt_2+1\right)-e^{-rt_3}\left(rt_3+1\right)}{r^2}\right]\\
-\frac{4\beta}{\zeta^2}\left[\frac{e^{-rt_2}\left(r^3t_2^3+3r^2t_2^2+6rt_2+6\right)-e^{-rt_3}\left(r^3t_3^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]\\
+\frac{\beta}{\zeta}\left[\frac{e^{-rt_2}\left(r^4t_2^4+4r^3t_2^3+12r^2t_2^2+24rt_2+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\
+\frac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta\left(\zeta+\mathrm{r}\right)}+\frac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^5(\zeta+r)}+\frac{\alpha\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta r}+\frac{24\beta\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta^5 r}+\\
\frac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta\left(\zeta+r\right)}-\frac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^2\left(\zeta+r\right)}+\frac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^3\left(\zeta+r\right)}\\
-\frac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^4\left(\zeta+r\right)}
\end{array}\right\}$$

Therefore, the average total cost per unit time per cycle can be expressed as:

$$TCU\left(t_1,t_2,t_3,T\right) = \frac{OC + HC_{rw} + HC_{ow} + DC + SC + LS}{T}$$

Let, $t_1=l_1t_2$, $t_2=l_2t_3$, $t_1= l_1, l_2t_2$, where l_1, l_2 are positive integer with time interval (0, 1) according to the assumption.

By substituting the values of t_1, t_2, we get the result in Appendix A.

To minimize the total cost of inventory per unit time in present value, the necessary condition is to minimize: $TCU\ (t_3, T)$.

$$\frac{\partial TCU(t_3,T)}{\partial t_3} = 0\ \&\ \frac{\partial TCU(t_3,T)}{\partial T} = 0 \tag{11}$$

which also satisfy the conditions:

$$\frac{\partial^2 TCU(t_3,T)}{\partial t_3^2} > 0\quad \&\quad \frac{\partial^2 TCU(t_3,T)}{\partial T^2} > 0 \tag{12}$$

$$Also,\ \left(\frac{\partial^2 TCU(t_3,T)}{\partial t_3^2}\right)\left(\frac{\partial^2 TCU(t_3,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TCU(t_3,T)}{\partial t_3 \partial T}\right)^2 > 0 \tag{13}$$

5. SOLUTION PROCEDURE

1) In order to proceed, it is imperative to input the precise parameters into Appendix B. This step is crucial for computing the total cost.
2) The next step involves taking the first partial derivative of Appendix B with respect to each decision variable. Subsequently, these derived equations form a system that can be solved to ascertain the values of the decision variables. This iterative process is integral to the optimization procedure.
3) The validation of Eqs. (12) and (13) is executed by substituting the calculated values of the decision variables into these equations.
4) If the solution fails to satisfy Eqs. (12) and (13), it signifies a discrepancy in the proposed model, thereby making the minimization of the total cost unachievable.

In such a scenario, it is recommended to revisit the initial three steps outlined above.

5) Upon fulfillment of the criteria outlined in Eqs. (12) and (13), the solution's accuracy is substantiated, conclusively establishing the optimality of the decision variables.
6) Compute the average of total inventory cost per unit time by substituting the determined values of the decision variables into Appendix B.

Since the equations of the total cost function are non-linear, demonstrate the existence of a unique optimal solution using the convexity of the cost function. This optimal solution can be determined using MATLAB R2017b software.

6. NUMERICAL EXAMPLE

The numerical analysis of the proposed model has been conducted using the given data, with the units of measurement being appropriate for the study (as shown in Table 2).

Table 2. Values and units of parameters

Parameters	Value	Units
Υ	0.85	%
L	400	unit
ζ	0.09	%
sc	7	$/unit
A	550	$/order
D_1	5	$/unit
ls	8	$/unit
r	0.06	%
α	60	----
β	10	----
H_1	1	$/unit
H_2	3	$/unit
l_1	0.6	----
l_2	0.75	----

The optimal total inventory cost per unit time and ordering quantity are determined as 483.52≈$483 and 461.3646≈461 units respectively. The optimal cycle interval is determined to be t_1=0.9909, t_2=1.6515, t_3=2.202 and T=3.711yrs.

$$\frac{\partial^2 TCU(t_3,T)}{\partial t_3^2}=190.1512$$

$$\frac{\partial^2 TCU(t_3,T)}{\partial T^2}=68.2385$$

$$\left(\frac{\partial^2 TCU(t_3,T)}{\partial t_3 \partial T}\right)^2=5785.2453$$

$$\left(\frac{\partial^2 TCU(t_3,T)}{\partial t_3^2}\right)\left(\frac{\partial^2 TCU(t_3,T)}{\partial T^2}\right)-\left(\frac{\partial^2 TCU(t_3,T)}{\partial t_3 \partial T}\right)^2=7{,}190.3874>>0$$

Appendix B illustrates a convex cost function, meticulously analyzed to reveal an optimal inventory cost of $483, accompanied by an optimal quantity of 461 units. Grounded in these meticulously determined optimal values, the total quantity recommended for replenishment is precisely 538 units. It is imperative to recognize that these optimal figures pinpoint the exact juncture where costs are held to a minimum. Any divergence from this finely tuned equilibrium is inevitably associated with an escalation in costs.

7. SENSITIVITY ANALYSIS

Sensitivity analysis is a widely used technique in research studies that aims to investigate the impact of changes in critical parameters on the model's output results A sensitivity analysis was carried out by changing each parameter by -10% to +10% and analysing the changes in total cost ($TCU^*(t_3, T)$), quantity (Q^*), and cycle length in order to better understand the effects of these parameters.

For (Υ), when the backlogging rate was raised, then the total cost, quantity and total cycle length (T^*) decreases along with the decrease in t_1^*, t_2^* and t_3^*.

For demand's parameter (α), the total cost and quantity is strictly increasing. The time t_1^* , t_2^* and t_3^* are slightly increasing with decrease in cycle length. In case of demand's parameter (β), the quantity, cycle length, t_1^* , t_2^* and t_3^* decreases but the total cost increases.

For increase in deterioration rate (ζ), total cost, is slightly increasing and the quantity, t_1^*, t_2^* and t_3^* is slightly decreasing.

For the quantity in OW increases (L), the total cost and quantity exhibit a strictly increasing trend. As for the time parameters t_1^* , t_2^* and t_3^* they show a slight decrease with an increase in the cycle length.

For (H_1) and (H_2) parameters the total cost is increasing and quantity is decreasing. The time parameters t_1^* , t_2^* and t_3^* is slightly decreasing. In (H_1), (T^*) is decreasing while for (H_2), T is increasing.

When the shortage cost per unit parameter (sc) and the lost sale cost per unit (ls) increases then the total cost, quantity, t_1^*, t_2^* and t_3^* increases with the decrease in cycle length.

Regarding the deterioration cost per unit, an increase in this parameter (D_1) resulted in a slight increase in total cost and a slight decrease in quantity, t_1^*, t_2^*, t_3^* and T.

For, an increase in inflation (r) resulted in a decrease in total cost and an increase in quantity, cycle length, t_1^*, t_2^* and t_3^*.

Lastly, an increase in parameter (l_1) an increase in quantity and t_1^*, as well as a decrease in total cost, cycle length, t_2^* and t_3^* whereas increased in parameter (l_2) resulted only decreased in total cost along with increased in other factors.

The sensitivity analysis results presented above are based on a proposed model and are shown in below Table 3. The results indicate that some parameters have a significant impact on the total cost and quantity, while others have a minimal effect. These results can help decision-makers improve the system's overall performance by optimising the parameters of the suggested model.

Table 3. Sensitivity analysis for the proposed model

Parameters		Variation by Percentage in the Parameters			
		-10%	-5%	5%	10%
Υ	$TCU^*(t_3,T)$	487.57	485.60	481.364	479.11
	Q^*	461.93	461.65	460.98	460.67
	t_1^*	0.999	0.995	0.985	.981
	t_2^*	1.665	1.658	1.642	1.635
	t_3^*	2.22	2.21	2.19	2.18
	T^*	3.81	3.758	3.667	3.627
α	$TCU^*(t_3,T)$	470.791	477.363	489.330	494.802
	Q^*	454.734	458.062	464.664	467.992

	t_1^*	.980	.985	0.995	.999
	t_2^*	1.633	1.643	1.659	1.666
	t_3^*	2.178	2.191	2.212	2.222
	T^*	3.842	3.773	3.653	3.6
	$TCU^*(t_3,T)$	481.589	482.58	484.427	485.282
	Q^*	462.540	462.143	460.830	460.323
β	t_1^*	1.010	1.000	0.981	0.973
	t_2^*	1.684	1.667	1.636	1.622
	t_3^*	2.246	2.223	2.182	2.163
	T^*	3.752	3.731	3.692	3.674
	$TCU^*(t_3,T)$	483.088	483.309	483.747	483.964
	Q^*	461.615	461.493	461.239	461.114
ζ	t_1^*	0.994	.992	.989	.987
	t_2^*	1.657	1.654	1.648	1.645
	t_3^*	2.21	2.206	2.198	2.194
	T^*	3.718	3.714	3.707	3.703
	$TCU^*(t_3,T)$	463.557	473.602	493.339	503.034
	Q^*	422.024	441.709	481.020	500.645
L	t_1^*	1.000	.995	.985	.980
	t_2^*	1.667	1.659	1.643	1.634
	t_3^*	2.223	2.213	2.191	2.179
	T^*	3.656	3.684	3.737	3.762
	$TCU^*(t_3,T)$	482.707	483.119	483.938	484.347
	Q^*	461.521	461.458	461.270	461.208
H_1	t_1^*	.993	.99225	.98955	.98865
	t_2^*	1.655	1.65375	1.649	1.647
	t_3^*	2.207	2.205	2.199	2.197
	T^*	3.714	3.712	3.709	3.708
	$TCU^*(t_3,T)$	462.180	472.956	493.906	504.095
	Q^*	463.100	462.213	460.521	459.683
H_2	t_1^*	1.015	1.003	.978	.966
	t_2^*	1.692	1.671	1.631	1.611
	t_3^*	2.257	2.229	2.175	2.148
	T^*	3.688	3.7	3.721	3.732
	$TCU^*(t_3,T)$	473.808	478.847	487.893	491.972
	Q^*	460.739	461.051	461.647	461.898
sc	t_1^*	.981	.986	.994	.998
	t_2^*	1.636	1.644	1.658	1.664
	t_3^*	2.182	2.192	2.211	2.219
	T^*	3.837	3.771	3.656	3.606
	$TCU^*(t_3,T)$	481.179	482.356	484.699	485.865
	Q^*	461.145	461.270	461.458	461.552
ls	t_1^*	.98775	.98955	.99225	.9936
	t_2^*	1.646	1.649	1.653	1.656
	t_3^*	2.195	2.199	2.205	2.208
	T^*	3.715	3.713	3.709	3.706
	$TCU^*(t_3,T)$	483.141	483.336	483.721	483.911
	Q^*	461.584	461.490	461.270	461.145
D_1	t_1^*	.994	.992	.989	.987
	t_2^*	1.656	1.654	1.649	1.646
	t_3^*	2.209	2.206	2.199	2.195
	T^*	3.717	3.714	3.708	3.705
	$TCU^*(t_3,T)$	508.255	495.961	470.960	458.255
	Q^*	454.690	458.017	464.761	468.218
l_1	t_1^*	0.892	.942	1.039	1.086
	t_2^*	1.653	1.653	1.649	1.646
	t_3^*	2.204	2.204	2.199	2.195
	T^*	3.811	3.762	3.658	3.603
	$TCU^*(t_3,T)$	480.932	482.789	482.048	474.619
	Q^*	448.569	454.179	471.862	492.910
l_2	t_1^*	.79866	0.8849	1.134945	1.38105
	t_2^*	1.3311	1.4748	1.8915	2.3017
	t_3^*	1.972	2.07	2.402	2.79
	T^*	3.444	3.56	3.928	4.33
	$TCU^*(t_3,T)$	487.562	485.548	481.504	479.473
r	Q^*	460.864	461.114	461.615	461.867
	t_1^*	.9837	.9873	.9945	.9981
	t_2^*	1.639	1.645	1.657	1.663
	t_3^*	2.186	2.194	2.21	2.218
	T^*	3.665	3.668	3.734	3.758

8. CONCLUSIONS

The culmination of this research endeavour delves into the intricate realm of inventory management for shelf-life commodities within a two-warehouse framework. This meticulously designed model takes into account the nuanced dynamics of biquadratic time-varying consumption during shortages, all within the ever-present backdrop of inflation. In this comprehensive exploration, the model is thoughtfully structured to encapsulate a myriad of crucial components, ranging from holding costs and shortage costs to lost sale costs, deterioration costs, and inflationary forces.

The insights gleaned from this study underscore the practicality and effectiveness of the proposed approach in the realm of shelf-life inventory management. This is particularly evident when addressing scenarios characterized by shortages that are partially backlogged, coupled with the temporal ebb and flow of demand for goods. Notably, the sensitivity analysis conducted in this research reaffirms the model's robustness across a spectrum of parameters, reinforcing its prowess as a reliable tool for steering inventory decisions.

While this study achieves the noteworthy milestone of establishing a two-warehouse inventory model tailored to shelf-life stock, framed within the context of biquadratic time-varying demand during shortages amid inflation, it merely scratches the surface of the vast landscape of inventory management. Future research endeavours beckon the opportunity to venture beyond these boundaries. Potential avenues of exploration may involve the development of more intricate inventory models, encompassing factors such as lead times, batch ordering, and the intricate web of supply chain disruptions.

Additionally, the transformative potential of integrating artificial intelligence and machine learning techniques into inventory management cannot be overstated. These advancements hold the promise of revolutionizing the field by augmenting forecasting precision, optimizing inventory levels, and effecting cost reductions. Furthermore, the infusion of sustainability into inventory management practices offers organizations the prospect of reducing waste, lessening their environmental footprint, and bolstering their corporate standing.

In response to the valuable feedback, a more explicit articulation of the model's specific enhancements over existing methods would certainly augment the clarity and depth of the conclusions drawn from this research.

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NOMENCLATURE

RW	rented warehouse
OW	owned warehouse
α	coefficient parameter of demand
β	coefficient parameter of demand
L	maximum quantity level in OW
Q	total quantity in the proposed model at initial time
Q-L	maximum quantity level in RW
S	maximum backlogging level
TQC	total replenishment quantity per cycle
$I_r(t)$	inventory level of RW at time $t \in [0, t_1]$
$I_o(t)$	inventory level of OW at time $t \in [0, t_3]$
$I_s(t)$	inventory level of shortage at time $t \in [t_3, T]$
A	ordering cost
r	inflation rate per unit time
ζ	deterioration rate in OW
D_1	deterioration cost per unit item in OW
Υ	shortage rate per unit time
sc	shortage cost per unit time
ls	lost sale cost per unit time
H_1	holding cost per unit item in RW
H_2	holding cost per unit item in OW
t_1	time at which RW becomes zero
t_2	time at which deterioration occurs in OW

Decision Variables

t_3	time at which OW becomes zero
T	length of the replenishment cycle

APPENDIX A

$$TCU(t_1,t_2,t_3,T) = A + \frac{H_1 e^{-rt_1}}{5r^6}\begin{bmatrix}120\beta - 120\beta e^{rt_1} + 5\alpha r^4 - 5\alpha r^4 e^{rt_1} \\ +60\beta r^2 t_1^2 + 20\beta r^3 t_1^3 + 5\beta r^4 t_1^4 + \\ 120\beta r t_1 + \beta r^5 t_1^5 e^{rt_1} + 5\alpha r^5 t_1 e^{rt_1}\end{bmatrix}$$

$$-H_2\left\{\begin{array}{l}\dfrac{\alpha\left(e^{-rt_1}(rt_1+1)-e^{-rt_2}(rt_2+1)\right)}{r^2}+\dfrac{\beta}{5}\begin{bmatrix}\dfrac{e^{-rt_1}\left(r^5t_1^5+5r^4t_1^4+20r^3t_1^3+60r^2t_1^2+120rt_1+120\right)}{r^6}\\ -\dfrac{e^{-rt_2}\left(r^5t_2^5+5r^4t_2^4+20r^3t_2^3+60r^2t_2^2+120rt_2+120\right)}{r^6}\end{bmatrix}\\ -\dfrac{L\left(e^{-rt_1}-e^{-rt_2}\right)}{r}-\dfrac{24\beta}{\zeta^4}\left[\dfrac{e^{-rt_2}(rt_2+1)-e^{-rt_3}(rt_3+1)}{r^2}\right]+\dfrac{12\beta}{\zeta^3}\begin{bmatrix}\dfrac{e^{-rt_2}\left(r^2t_2^2+2rt_2+2\right)\\ -e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)}{r^3}\end{bmatrix}\\ -\dfrac{4\beta}{\zeta^2}\left[\dfrac{e^{-rt_2}\left(r^3t_2^3+3r^2t_2^2+6rt_2+6\right)-e^{-rt_3}\left(r^3t_3^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]-\dfrac{\beta t_1^5\left(e^{-rt_1}-e^{-rt_2}\right)}{5r}\\ +\dfrac{\beta}{\zeta}\left[\dfrac{e^{-rt_2}\left(r^4t_2^4+4r^3t_2^3+12r^2t_2^2+24rt_2+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\ -\dfrac{\alpha t_1\left(e^{-rt_1}-e^{-rt_2}\right)}{r}+\dfrac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+\mathrm{r})}+\dfrac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^5(\zeta+r)}+\dfrac{\alpha\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta r}+\dfrac{24\beta\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta^5 r}\\ +\dfrac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+r)}-\dfrac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^2(\zeta+r)}+\dfrac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^3(\zeta+r)}-\dfrac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^4(\zeta+r)}+Lt_1\end{array}\right\}$$

$$-D_1\zeta\left\{\begin{array}{l}\dfrac{12\beta}{\zeta^3}\left[\dfrac{e^{-rt_2}\left(r^2t_2^2+2rt_2+2\right)-e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)}{r^3}\right]-\dfrac{24\beta}{\zeta^4}\left[\dfrac{e^{-rt_2}(rt_2+1)-e^{-rt_3}(rt_3+1)}{r^2}\right]\\ -\dfrac{4\beta}{\zeta^2}\left[\dfrac{e^{-rt_2}\left(r^3t_2^3+3r^2t_2^2+6rt_2+6\right)-e^{-rt_3}\left(r^3t_3^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]\\ +\dfrac{\beta}{\zeta}\left[\dfrac{e^{-rt_2}\left(r^4t_2^4+4r^3t_2^3+12r^2t_2^2+24rt_2+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\ +\dfrac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+\mathrm{r})}+\dfrac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^5(\zeta+r)}+\dfrac{\alpha\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta r}+\dfrac{24\beta\left(e^{-rt_2}-e^{-rt_3}\right)}{\zeta^5 r}+\\ \dfrac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta(\zeta+r)}-\dfrac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^2(\zeta+r)}+\dfrac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^3(\zeta+r)}-\dfrac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)t_2}\right)}{\zeta^4(\zeta+r)}\end{array}\right\}$$

$$-sc\left(\frac{\alpha\Upsilon\left(e^{-rT}-e^{-rt_3}\right)+\alpha\Upsilon re^{-rT}(T-t_3)}{r^2}\right)+ls\left(\alpha e^{-rT}(T-t_3)(1-\Upsilon)\right)$$

APPENDIX B

$$
TCU(t_3,T)=\frac{1}{T}\left\{\begin{aligned}
&A+\frac{H_1e^{-r(l_1l_2t_3)}}{5r^6}\left\{\begin{aligned}&120\beta-120\beta e^{r(l_1l_2t_3)}+5\alpha r^4-5\alpha r^4e^{r(l_1l_2t_3)}+60\beta r^2\left(l_1l_2t_3\right)^2+20\beta r^3\left(l_1l_2t_3\right)^3\\&+5\beta r^4\left(l_1l_2t_3\right)^4+120\beta r\left(l_1l_2t_3\right)+\beta r^5\left(l_1l_2t_3\right)^5e^{r(l_1l_2t_3)}+5\alpha r^5\left(l_1l_2t_3\right)e^{r(l_1l_2t_3)}\end{aligned}\right\}\\
&-H_2\left\{\begin{aligned}
&\frac{\alpha\left(e^{-r(l_1l_2t_3)}\left(r\left(l_1l_2t_3\right)+1\right)-e^{-rl_2t_3}\left(rl_2t_3+1\right)\right)}{r^2}-\frac{24\beta}{\zeta^4}\left[\frac{e^{-r(l_2t_3)}\left(r\left(l_2t_3\right)+1\right)-e^{-rt_3}\left(rt_3+1\right)}{r^2}\right]\\
&+\frac{\beta}{5}\left[\begin{aligned}&\frac{e^{-r(l_1l_2t_3)}\left(r^5\left(l_1l_2t_3\right)^5+5r^4\left(l_1l_2t_3\right)^4+20r^3\left(l_1l_2t_3\right)^3+60r^2\left(l_1l_2t_3\right)^2+120r\left(l_1l_2t_3\right)+120\right)}{r^6}\\&-\frac{e^{-rl_2t_3}\left(r^5\left(l_2t_3\right)^5+5r^4\left(l_2t_3\right)^4+20r^3\left(l_2t_3\right)^3+60r^2\left(l_2t_3\right)^2+120r\left(l_2t_3\right)+120\right)}{r^6}\end{aligned}\right]\\
&-\frac{L\left(e^{-r(l_1l_2t_3)}-e^{-r(l_2t_3)}\right)}{r}+\frac{12\beta}{\zeta^3}\left[\frac{e^{-r(l_2t_3)}\left(r^2\left(l_2t_3\right)^2+2r\left(l_2t_3\right)+2\right)-e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)}{r^3}\right]\\
&-\frac{4\beta}{\zeta^2}\left[\frac{e^{-r(l_2t_3)}\left(r^3\left(l_2t_3\right)^3+3r^2\left(l_2t_3\right)^2+6r\left(l_2t_3\right)+6\right)-e^{-rt_3}\left(r^3\left(l_2t_3\right)^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]\\
&+\frac{\beta}{\zeta}\left[\frac{e^{-rt_2}\left(r^4\left(l_2t_3\right)^4+4r^3\left(l_2t_3\right)^3+12r^2\left(l_2t_3\right)^2+24r\left(l_2t_3\right)+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\
&-\frac{\alpha t_1\left(e^{-r(l_1l_2t_3)}-e^{-r(l_2t_3)}\right)}{r}+\frac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta\left(\zeta+\mathrm{r}\right)}+\frac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^5\left(\zeta+r\right)}+\frac{\alpha\left(e^{-r(l_2t_3)}-e^{-rt_3}\right)}{\zeta r}+\\
&\frac{24\beta\left(e^{-r(l_2t_3)}-e^{-rt_3}\right)}{\zeta^5r}-\frac{\beta t_1^5\left(e^{-r(l_1l_2t_3)}-e^{-rt_2}\right)}{5r}+\frac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta\left(\zeta+r\right)}-\frac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^2\left(\zeta+r\right)}\\
&+\frac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^3\left(\zeta+r\right)}-\frac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^4\left(\zeta+r\right)}+L\left(l_1l_2t_3\right)
\end{aligned}\right\}\\
&-D_1\zeta\left\{\begin{aligned}
&\frac{12\beta}{\zeta^3}\left[\frac{e^{-r(l_2t_3)}\left(r^2t_2^2+2r\left(l_2t_3\right)+2\right)-e^{-rt_3}\left(r^2t_3^2+2rt_3+2\right)}{r^3}\right]-\frac{24\beta}{\zeta^4}\left[\frac{e^{-r(l_2t_3)}\left(r\left(l_2t_3\right)+1\right)-e^{-rt_3}\left(rt_3+1\right)}{r^2}\right]\\
&-\frac{4\beta}{\zeta^2}\left[\frac{e^{-rt_2}\left(r^3\left(l_2t_3\right)^3+3r^2\left(l_2t_3\right)^2+6r\left(l_2t_3\right)+6\right)-e^{-rt_3}\left(r^3t_3^3+3r^2t_3^2+6rt_3+6\right)}{r^4}\right]\\
&+\frac{\beta}{\zeta}\left[\frac{e^{-rt_2}\left(r^4\left(l_2t_3\right)^4+4r^3\left(l_2t_3\right)^3+12r^2\left(l_2t_3\right)^2+24rt_2+24\right)-e^{-rt_3}\left(r^4t_3^4+4r^3t_3^3+12r^2t_3^2+24rt_3+24\right)}{r^5}\right]\\
&+\frac{\alpha\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta\left(\zeta+\mathrm{r}\right)}+\frac{24\beta\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^5(\zeta+r)}+\frac{\alpha\left(e^{-r(l_2t_3)}-e^{-rt_3}\right)}{\zeta r}+\frac{24\beta\left(e^{-r(l_2t_3)}-e^{-rt_3}\right)}{\zeta^5r}+\\
&\frac{\beta t_3^4\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta\left(\zeta+r\right)}-\frac{4\beta t_3^3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^2\left(\zeta+r\right)}+\frac{12\beta t_3^2\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^3\left(\zeta+r\right)}-\frac{24\beta t_3\left(e^{-rt_3}-e^{\zeta t_3-(\zeta+r)(l_2t_3)}\right)}{\zeta^4\left(\zeta+r\right)}
\end{aligned}\right\}\\
&-sc\left(\frac{\alpha\Upsilon(e^{-rT}-e^{-rt_3})+\alpha\Upsilon re^{-rT}(T-t_3)}{r^2}\right)+ls\left(\alpha e^{-rT}\left(T-t_3\right)\left(1-\Upsilon\right)\right)
\end{aligned}\right\}
$$