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Results of Fourth-Order Differential Superordination and Subordination for Univalent Functions Defined by Integral Operator

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ABSTRACT

Exploring some characteristics of differential subordination and superordination of analytic univalent functions in an open unit disc is the aim of this work and additionally, we have the form's normalized Taylor-Maclaurin series: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. It also aims to clarify the results of the sandwich. By utilizing the integral operator's properties to examine forth-order differential subordination and superordination of analytic univalent functions, some fascinating results are found and explore forth-order subordinations and superordinations in respect to the convolution. Ultimately, we acquired multiple outcomes concerning fourth-order sandwich theorems inside the open unit disk.

1. INTRODUCTION

Suppose that $\mathbb{C}$ be a complex plane and $R = R(\mathbb{U})$ be the classof functions which are analytic in the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$. Regarding an integer number that is positive n and $a \in \mathbb{C}$, we assume that $R[a, n]$ be the R subclass made up of functions with the structure $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, (z \in \mathbb{U})$, and R_1=[1,1]. Let f and k are analytic in U, if there is a Schwarz function w in U such that $w(o)=o$, then we say that the function f is subordinate to k, or that k is superordinate to f, and $|w(z)| < 1$ (z∈U) where $f(z)=k(w(z))$. When that occurs, we write $f \prec k$ or $f(z) \prec k(z)$ $(z \in U)$.

Specifically, in cases when the function g is univalent in U, then $f \prec g$ if and only if $f(o)=g(o)$ and $f(U) \subset g(U)$.

Let Du represent the class of functions of the form [1]:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

Darweesh [2] presented and examined the new integral operator:

$$\mathcal{J}_{(\alpha,\beta)}: D_u \to D_u,$$

which is defined as follows:

$$\mathcal{J}_{(\alpha,\beta)} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(\alpha+n-1)}{(\ln(\beta))^{n-1}\Gamma(\alpha)} \right) a_n z^n,$$

where, $f \in D_u$, for $\alpha \in N$ and $\beta \geq 2$.

It is easily verified from Eq. (2), that

$$z\left(\mathcal{J}_{(\alpha,\beta)} f(z)\right)' = \alpha \mathcal{J}_{(\alpha+1,\beta)} f(z) - (\alpha - 1)\mathcal{J}_{(\alpha,\beta)} f(z). \quad (2)$$

In recent times, for example, a number of authors [3-15] have developed and talked about the idea of superordination and second-order differential subordination. In addition, a number of writers covered topics such as the theory of third-order differential subordination and superordination [16-19]. The theory of the second-order differential subordination in the open unit disk, first introduced by Miller and Mocanu [12], was expanded to the third instance by Antonino and Miller [16] in 2011. The third-order instance was extended to fourth-order differential subordination by Atshan et al. [20, 21], who also identified the features of functions g that fulfill the subsequent fourth-order differential subordination:

$$\phi(g(z), zg'(z), z^2 g''(z), z^3 g'''(z), z^4 g''''(z); z) \prec \mathfrak{h}(z),$$

where, h beanalytic univalent function in $\mathbb{U}$, g is $analytic\ function$ and $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$. Now, we identified characteristics of the function g that satisfy the subsequent fourth-order differential superordination after extending the third-order case to a fourth-order case:

$$h(z) \prec \phi(g(z), zg'(z), z^2 g''(z), z^3 g'''(z), z^4 g''''(z); z),$$

where, h be analytic univalent function in $\mathbb{U}$, g isanalytic function $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$. To prove ourmain result, weneed the basic conceptsin the theoryof fourth-order.

Definition (1) [16]: Let Q represent the collection of all univalent and analytic functions q on the set $\overline{\mathbb{U}} \setminus \mathcal{E}(q)$, where

$\mathcal{E}(\mathrm{q}) = \left\{\zeta \in \partial\mathbb{U}: \lim_{z\to\zeta} \mathrm{q}(z) = \infty\right\}$, such that, $\min|\mathrm{q}'(z)| = \rho > 0$ for $\zeta \in \overline{\mathbb{U}} \setminus \mathcal{E}(\mathrm{q})$. Furthermore, let Q(a) be the subclass of Q for which q(o)=a, where, Q(0)= Q_0, Q(1)= Q_1, and Q_1={q∈Q: q(0)=1}.

Definition (2) [20, 21]: Assume that h(z) is a univalent function in U and let $\phi: C^5 \times U \to C$. Assume that $g(z)$ is an analytic function in U that fulfills the fourth-order differential subordination listed below:

$$\phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z) \prec \mathfrak{h}(z), \quad (3)$$

$g(z)$ is therefore referred to as a differential subordination (3) solution. A dominating of the solutions of (3) is a univalent function q(z). or, simpler, a dominant $\mathrm{q}(z)$ *if* $g(z) \prec \mathrm{q}(z)$ for all $g(z)$ satisfying (3). A dominant $\check{\mathrm{q}}(z)$which satisfies $\check{\mathrm{q}}(z) \prec \mathrm{q}(z)$, the best dominant for all dominants is q(z) of (3).

Definition (3) [21]: Assume that h(z) is an analytic function in U and that $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$. Given $g(z)$ and:

$$\phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z),$$

Meet the requirements for the following univalent functions in U and fourth-order differential superordination:

$$\mathfrak{h}(z) \prec \phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z), \quad (4)$$

Thus, $g(z)$ is called a differential superordination (4) solution. An analytical position to put it another way, $q(z)$ is a subordinant of the solution to (4) if, for every $g(z)$ satisfying (4), $q(z) \prec g(z)$. The univalent subordinant $\hat{q}(z)$ that satisfies $\mathrm{q}(z) \prec \check{\mathrm{q}}(z)$for all subordinants of (4) is the finest subordinant. It is noted that up until U rotates, the optimal subordinant is unique.

Definition (4) [20, 21]: Consider a set Ω in C, q in Q, and n *in* N\{2}. Functions $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$ that meet the subsequent admissibility requirement are included in the class $\Psi_n[\Omega, \mathrm{q}]$of admissible functions:

$$\phi(u, v, x, y, g; \zeta) \notin \Omega,$$

whenever

$$\mathrm{u} = \mathrm{q}(\zeta),\ v = k\zeta\mathrm{q}'(\zeta),\ \mathcal{R}e\left\{\frac{x}{v} + 1\right\} \geqq k\mathcal{R}e\left\{\frac{\zeta\mathrm{q}''(\zeta)}{\mathrm{q}'(\zeta)} + 1\right\},$$

and

$$\mathcal{R}e\left\{\frac{y}{v}\right\} \geqq k^2\mathcal{R}e\left\{\frac{\zeta^2\mathrm{q}'''(\zeta)}{\mathrm{q}'(\zeta)}\right\},\ \mathcal{R}e\left\{\frac{g}{v}\right\} \geqq k^3\mathcal{R}e\left\{\frac{\zeta^3\mathrm{q}''''(\zeta)}{\mathrm{q}'(\zeta)}\right\},$$

where, z∈ $\mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathrm{q})$ and $k \geqq 3$.

Definition (5) [21]: Given a set Ω in C, $q(z) \in R[a, n]$ and $\mathrm{q}'(z) \neq 0$, functions $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$ which fulfill the following admissibility requirement are included in the class $\Psi'_n[\Omega, \mathrm{q}]$ of admissible functions:

$$\phi(u, v, x, y, g; \zeta) \in \Omega,$$

whenever

$$\mathrm{u} = \mathrm{q}(z),\ v = \frac{z\mathrm{q}'(z)}{m},\ \mathcal{R}e\left\{\frac{x}{v} + 1\right\} \geqq \frac{1}{m}\mathcal{R}e\left\{\frac{z\mathrm{q}''(z)}{\mathrm{q}'(z)} + 1\right\},$$

and

$$\mathcal{R}e\left\{\frac{y}{v}\right\} \leqq \frac{1}{m^2}\mathcal{R}e\left\{\frac{z^2\mathrm{q}'''(z)}{\mathrm{q}'(z)}\right\},$$
$$\mathcal{R}e\left\{\frac{g}{v}\right\} \geqq \frac{1}{m^3}\mathcal{R}e\left\{\frac{z^3\mathrm{q}''''(z)}{\mathrm{q}'(z)}\right\},$$

where, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ and $m \geqq n \geqq 3$.

The following lemma contains the fundamental outcome for the theory of fourth-order differential subordination.

Lemma (1) [6]: Assume that $g \in R[a, n]$ and that $n \in N \setminus \{2\}$ and $q \in Q(a)$ both meet the requirements listed below.

$$\mathcal{R}e\left\{\frac{\zeta^2\mathrm{q}'''(\zeta)}{\mathrm{q}'(\zeta)}\right\} \geqq 0, \text{ and } \left|\frac{z^2p''(z)}{\mathrm{q}'(\zeta)}\right| \leqq k^2,$$

where, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathrm{q})$ and $k \geqq n$. *If* Ω is a set in $\mathbb{C}$, $\phi \in \Psi_n[\Omega, \mathrm{q}]$ and:

$$\phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z) \subset \Omega,$$

then

$$g(z) \prec \mathrm{q}(z), \qquad (z \in \mathbb{U}).$$

The fundamental outcome of the idea of fourth-order differential superordination is found in the following lemma.

Lemma (2) [21]: The basic result in the theory of fourth-order differential subordination is found in the following lemma.

$$\mathrm{q}(z) \in R[a, n] \text{ with } \phi \in \Psi'_n[\Omega, \mathrm{q}].$$

If $\phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z)$, is univalent in U and $g(z) \in Q(a)$ fulfills the requirements listed below for admissibility:

$$\mathcal{R}e\left\{\frac{z^2\mathrm{q}'''(z)}{\mathrm{q}'(\zeta)}\right\} \geqq 0, \text{ and } \left|\frac{z^2p''(z)}{\mathrm{q}'(\zeta)}\right| \leqq \frac{1}{m^2},$$

where, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ and $m \geqq n \geqq 3$, then

$$\Omega \subset \{\phi(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z): z \in \mathbb{U}\},$$

implies that

$$\mathrm{q}(z) \prec g(z), (z \in \mathbb{U}).$$

This concept's major goal is to identify the necessary requirements that certain normalized analytic functions f must meet in order to satisfy:

$$\mathrm{q}_1(z) \prec \mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathrm{q}_2(z),$$
$$\mathrm{q}_1(z) \prec z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathrm{q}_2(z),$$

where, $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $\mathrm{q}_1(0) = \mathrm{q}_2(0) = 1$.

2. FOURTH-ORDER DIFFERENTIAL SUBORDINATION RESULTS USING $\mathcal{J}_{(\alpha,\beta)}f(z)$

Prior to proving the differential theorems connected to $\mathcal{J}_{(\alpha,\beta)}$ provided by Eq. (1), we establish the class of admissible functions that follow.

Definition (6): Let $\mathfrak{q} \in \mathcal{Q}_0 \cap \mathcal{J}_0$ Ω, be a set in C. Functions $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$, that meet the following admissibility requirements make up the class $\mathrm{M}_i[\Omega, \mathfrak{q}]$ of admissible functions:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

when

$$u = \mathfrak{q}(\xi), \quad v = \frac{k\xi\mathfrak{q}'(\xi)+(\alpha-1)\mathfrak{q}(\xi)}{\alpha},$$
$$\mathcal{R}e\left(\frac{x+\alpha(2v+(\alpha-1)u)}{\alpha(u+\alpha(v-u)}\right) \geqq k\mathcal{R}e\left\{\frac{\zeta\mathfrak{q}''(\zeta)}{\mathfrak{q}'(\zeta)}+1\right\},$$
$$\mathcal{R}e\left(\frac{y+\alpha(3x+u)+2\alpha^2v-(v+u)}{\alpha^2(v-u)+\alpha^3u}\right) \geqq k^2\mathcal{R}e\left\{\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right\}$$

and

$$\mathcal{R}e\left(\frac{g + 2\alpha(2y + 3x) - (y + 4x) + 5\alpha^2x + (\alpha-1)^4((2\alpha^2-1)^2v + u)}{\alpha^3(v + (\alpha-1)u)}\right) \geqq k^3\mathcal{R}e\left(\frac{\zeta^3\mathfrak{q}''''(\zeta)}{\mathfrak{q}'(\zeta)}\right),$$

where, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathfrak{q})$ and $k \geqq 3$.

Theorem 1: Let $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$. If the following criteria are met and f and $\mathfrak{q}$ both fall inside D_u and $\mathcal{Q}_0$:

$$\mathcal{R}e\left(\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right) \geqq 0, \quad \left|\frac{\mathcal{I}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(\zeta)}\right| \leqq k^2, \tag{5}$$

and

$$\left\{\phi\left(\mathcal{I}_{(\alpha,\beta)}f(z), \mathcal{I}_{(\alpha+1,\beta)}f(z), \mathcal{I}_{(\alpha+2,\beta)}f(z), \mathcal{I}_{(\alpha+3,\beta)}f(z), \mathcal{I}_{(\alpha+4,\beta)}f(z)\right)\right\} \subset \Omega, \tag{6}$$

then

$$\mathcal{I}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z) \qquad (z \in \mathbb{U}).$$

Proof: Give the definition of the analytic function $\mathfrak{p}(z) in\ \mathbb{U}$.

$$\mathfrak{p}(z) = \mathcal{I}_{(\alpha,\beta)}f(z) \quad (z \in \mathbb{U}). \tag{7}$$

Next, employing Eq. (3) and differentiating Eq. (7) with regard to *z*, we have

$$\mathcal{I}_{(\alpha+1,\beta)}f(z) = \frac{z\mathfrak{p}'(z)+(\alpha-1)\mathfrak{p}(z)}{\alpha}. \tag{8}$$

Further computations show that

$$\mathcal{I}_{(\alpha+2,\beta)}f(z) = \frac{z^2\mathfrak{p}''(z)+(2\alpha-1)z\mathfrak{p}'(z)+(\alpha-1)^2\mathfrak{p}(z)}{\alpha^2} \tag{9}$$

$$\mathcal{I}_{(\alpha+3,\beta)}f(z) = \frac{z^3\mathfrak{p}^{(3)}(z)+3\alpha z^2\mathfrak{p}''(z)+(2\alpha^2-1)z\mathfrak{p}'(z)+(\alpha-1)^3\mathfrak{p}(z)}{\alpha^3}, \tag{10}$$

and

$$\mathcal{I}_{(\alpha+4,\beta)}f(z) = \frac{z^4\mathfrak{p}^{(4)}(z)+2(2\alpha+1)z^3\mathfrak{p}^{(3)}(z)+(5\alpha^2+6\alpha-4)z^2\mathfrak{p}''(z)}{\alpha^4} + \frac{(2\alpha^2-1)^2(\alpha-1)^4z\mathfrak{p}'(z)+(\alpha-1)^4\mathfrak{p}(z)}{\alpha^4}. \tag{11}$$

We now define the transformation $\mathbb{C}^5$ to $\mathbb{C}$,

$$u(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \mathfrak{r}, \quad v(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{s}+(\alpha-1)\mathfrak{r}}{\alpha},$$
$$x(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{t}+(2\alpha-1)\mathfrak{s}+(\alpha-1)^2\mathfrak{r}}{\alpha^2}, \tag{12}$$
$$y(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{w+3\alpha\mathfrak{t}+(2\alpha^2-1)\mathfrak{s}+(\alpha-1)^3\mathfrak{r}}{\alpha^3},$$

and

$$g(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{b}+2(2\alpha+1)w+(5\alpha^2+6\alpha-4)\mathfrak{t}+(2\alpha^2-1)^2(\alpha-1)^4\mathfrak{s}+(\alpha-1)^4\mathfrak{r}}{\alpha^4}, \tag{13}$$

let

$$\chi(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b};z) = \phi(u,v,x,y,g;z) = \phi\left(\mathfrak{r}, \frac{\mathfrak{s}+(\alpha-1)\mathfrak{r}}{\alpha}, \frac{\mathfrak{t}+(2\alpha-1)\mathfrak{s}+(\alpha-1)^2\mathfrak{r}}{\alpha^2}, \frac{w+3\alpha\mathfrak{t}+(2\alpha^2-1)\mathfrak{s}+(\alpha-1)^3\mathfrak{r}}{\alpha^3}, \frac{\mathfrak{b}+2(2\alpha+1)w+(5\alpha^2+6\alpha-4)\mathfrak{t}+(2\alpha^2-1)^2(\alpha-1)^4\mathfrak{s}+(\alpha-1)^4\mathfrak{r}}{\alpha^4}; z\right). \tag{14}$$

Lemma (1) will be used in the proof. Eq. (14) gives us, using Eqs. (7)-(11), that

$$\chi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z\right) = \phi\left(\mathcal{I}_{(\alpha,\beta)}f(z), \mathcal{I}_{(\alpha+1,\beta)}f(z), \mathcal{I}_{(\alpha+2,\beta)}f(z), \mathcal{I}_{(\alpha+3,\beta)}f(z), \mathcal{I}_{(\alpha+4,\beta)}f(z)\right). \tag{15}$$

Thus, it follows that Eq. (6) becomes:

$$\chi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z\right) \in \Omega,$$

we observe that

$$\frac{\mathfrak{t}}{\mathfrak{s}} + 1 = \frac{x+\alpha(2v+(\alpha-1)u)}{\alpha(u+\alpha(v-u)}, \quad \frac{w}{\mathfrak{s}} = \frac{y+\alpha(3x+u)+2\alpha^2v-(v+u)}{\alpha^2(v-u)+\alpha^3u},$$

and

$$\frac{\mathfrak{b}}{\mathfrak{s}} = \frac{g + 2\alpha(2y + 3x) - (y + 4x) + 5\alpha^2x + (\alpha-1)^4((2\alpha^2-1)^2v + u)}{\alpha^3(v + (\alpha-1)u)}.$$

The admissibility condition for ϕ, in Definition (6) is therefore equal to the admissibility condition for χ in Definition (4) with $n = 3, for\ \chi \in \Psi_3[\Omega, \mathfrak{q}]$. Thus, applying Lemma (1) and (5), we get

$$\mathfrak{p}(z) = \mathcal{I}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z).$$

This concludes Theorem (1)'s proof.

The next corollary extends Theorem (1) to the situation where $\mathfrak{q}(z) \in \partial\mathbb{U}$ behavior is unknown.

Corollary (1): Assume that $\Omega \subset \mathbb{C}$ and that $q(z)$ is a univalent function in U such that $q(0) = 1$. For any $\rho \in (0,1)$, let $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$, where $\mathfrak{q}_\rho(z) = \mathfrak{q}(\rho z)$. If both $\mathfrak{q}_\rho$ and $f \in D_u$ meet the requirements listed below:

$$\mathcal{R}e\left(\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right) \geqq 0, \left|\frac{\mathcal{I}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(\zeta)}\right| \leqq k^2, \left(z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathfrak{q}_\rho) \text{ and } k \geqq n\right) \tag{16}$$

and

$$\phi\begin{pmatrix}\mathcal{J}_{(\alpha,\beta)}f(z),\mathcal{J}_{(\alpha+1,\beta)}f(z),\mathcal{J}_{(\alpha+2,\beta)}f(z),\\ \mathcal{J}_{(\alpha+3,\beta)}f(z),\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{pmatrix}\prec \mathfrak{h}(z), \qquad (17)$$

then

$$\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}(z)\ (z\in\mathbb{U}).$$

Proof: By using Theorem (1), yield

$$\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}_\rho(z)\ (z\in\mathbb{U}),$$

then, we get the outcome from

$$\mathfrak{q}_\rho(z)\prec \mathfrak{q}(z) \quad (z\in\mathbb{U}).$$

This concludes Corollary (1)'s evidence.

$\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of U on to Ω if $\Omega\neq C$ is a simply connected domain. In this instance, a $M_i[\mathfrak{h},\mathfrak{q}]$ is written for the class $M_i[h(\mathbb{U}),\mathfrak{q}]$. The following two results are immediate consequence of Theorem (1) and Corollary (1).

Theorem (2): Let $\phi\in M_i[\mathfrak{h},q]$. Assuming that $f\in D_u$ and $\mathfrak{q}\in\mathcal{Q}_0$ meet condition (5), and

$$\mathcal{R}e\left(\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right)\geqq 0, \qquad \left|\frac{\mathcal{J}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(\zeta)}\right|\leqq k^2, \qquad (18)$$

$$\phi\begin{pmatrix}\mathcal{J}_{(\alpha,\beta)}f(z),\mathcal{J}_{(\alpha+1,\beta)}f(z),\mathcal{J}_{(\alpha+2,\beta)}f(z),\\ \mathcal{J}_{(\alpha+3,\beta)}f(z),\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{pmatrix}\prec \mathfrak{h}(z), \qquad (19)$$

then

$$\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}(z) \quad (z\in\mathbb{U}).$$

Corollary (2): Assume that $\Omega\subset\mathbb{C}$ and that $\mathfrak{q}$ is a univalent function in U such that $q(0)=1$. For any $\rho\in(0,1)$, let $\phi\in M_i[\mathfrak{h},\mathfrak{q}]$, where $\mathfrak{q}_\rho(z)=\mathfrak{q}(\rho z)$. If both $\mathfrak{q}_\rho$ and $f\in D_u$ meet the requirements (18):

$$\phi\left(\mathcal{J}_{(\alpha,\beta)}f(z),\mathcal{J}_{(\alpha+1,\beta)}f(z),\mathcal{J}_{(\alpha+2,\beta)}f(z),\mathcal{J}_{(\alpha+3,\beta)}f(z),\mathcal{J}_{(\alpha+4,\beta)}f(z)\right)\prec \mathfrak{h}(z), \qquad (20)$$

then

$$\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}(z)(z\in\mathbb{U}).$$

The best dominant of differential subordination is produced by the following finding (17).

Theorem (3): Assume that $h\ is\ a\ univalent\ function\ in$ U. Additionally, let

$$\phi\begin{pmatrix}\mathfrak{q}(z),\frac{z\mathfrak{q}'(z)+(\alpha-1)\mathfrak{q}(z)}{\alpha},\frac{z^2\mathfrak{q}''(z)+(2\alpha-1)z\mathfrak{q}'(z)+(\alpha-1)^2\mathfrak{q}(z)}{\alpha^2},\\ \frac{z^3\mathfrak{q}^{(3)}(z)+3\alpha z^2\mathfrak{q}''(z)+(2\alpha^2-1)z\mathfrak{q}'(z)+(\alpha-1)^3\mathfrak{q}(z)}{\alpha^3},\\ \frac{z^4\mathfrak{q}^{(4)}(z)+2(2\alpha+1)z^3\mathfrak{q}^{(3)}(z)+(5\alpha^2+6\alpha-4)z^2\mathfrak{q}''(z)}{\alpha^4}+\\ \frac{(2\alpha^2-1)^2(\alpha-1)^4z\mathfrak{q}'(z)+(\alpha-1)^4\mathfrak{q}(z)}{\alpha^4}.;z\end{pmatrix}=\mathfrak{h}(z), \qquad (21)$$

has a solution $q(z)$ that meets condition (5) because $q(0)=1$. If the condition (16) is met by f satisfying D_u, , and if $\phi\begin{pmatrix}\mathcal{J}_{(\alpha,\beta)}f(z),\mathcal{J}_{(\alpha+1,\beta)}f(z),\mathcal{J}_{(\alpha+2,\beta)}f(z),\\ \mathcal{J}_{(\alpha+3,\beta)}f(z),\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{pmatrix}$, is analytic in U, then

$$\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}(z) \quad (z\in\mathbb{U}).$$

where the best dominant is $q(z)$.

Proof: As may be shown from Theorem (1), $q(z)$ is a dominant of Eq. (17). Moreover, $q(z)$ is a solution of (17) since it satisfies (20). Consequently, all dominating will dominate $q(z)$. As a result, the best dominating is $q(z)$.

Definition (7): Let Ω a set in $\mathbb{C}$ and $\mathfrak{q}\in\mathcal{Q}_1\cap\mathcal{J}_1$. Functions $\phi:\mathbb{C}^5\times\mathbb{U}\to\mathbb{C}$, that meet the following admissibility requirements make up the class $M_{i,1}[\Omega,\mathfrak{q}]$ of admissible functions:

$$\phi(u,v,x,y,g;z)\notin\Omega,$$

when

$$u=\mathfrak{q}(\xi),$$
$$v=\frac{k\xi\mathfrak{q}'(\xi)+\alpha\mathfrak{q}(\xi)}{\alpha},$$
$$\mathcal{R}e\left(\frac{(x+v)+\alpha(3v+\alpha u)}{\alpha(v+\alpha u)}\right)\geqq k\mathcal{R}e\left\{\frac{\zeta\mathfrak{q}''(\zeta)}{\mathfrak{q}'(\zeta)}+1\right\},$$
$$\mathcal{R}e\left(\frac{(y-3x+v)+\alpha(3x+5v)+\alpha^2(2v+\alpha u}{\alpha^2(v+\alpha u)}\right)\geqq k^2\mathcal{R}e\left\{\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right\}$$

and

$$\mathcal{R}e\left(\frac{(g+v-5x)+\alpha^2(5x+7v)+2\alpha(4x+3v)+\alpha^3(2v+\alpha u)}{\alpha^3(v+\alpha u)}\right)\geqq k^3\mathcal{R}e\left(\frac{\zeta^3\mathfrak{q}''''(\zeta)}{\mathfrak{q}'(\zeta)}\right),$$

when, $z\in\mathbb{U},\zeta\in\partial\mathbb{U}\setminus\mathcal{E}(\mathfrak{q})$ and $k\geqq 3$.

Theorem (4): Explain how $\phi\in M_i[h,\mathfrak{q}]$. If both $\mathfrak{q}\in\mathcal{Q}_1$ and $f\in D_u$, and meet the requirements listed below,

$$\mathcal{R}e\left(\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right)\geqq 0, \quad \left|\frac{z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(\zeta)}\right|\leqq k^2, \qquad (22)$$

and

$$\left\{\phi\begin{pmatrix}z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z),z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z),z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z),\\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z),z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{pmatrix}\right\}\subset\Omega, \qquad (23)$$

then

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z)\prec \mathfrak{q}(z) \qquad (z\in\mathbb{U}).$$

Proof: Assume that the analytic function $p(z)$ in U is defined by

$$\mathfrak{p}(z)=z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z). \qquad (24)$$

From Eqs. (2) and (24), we have:

$$z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z)=\frac{z^2\mathfrak{p}'(z)+\alpha z\mathfrak{p}(z)}{\alpha}. \qquad (25)$$

By a similar argument, we get:

$$z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z) = \frac{z^3\mathfrak{p}''(z)+(2\alpha+1)z^2\mathfrak{p}'(z)+\alpha^2 z\mathfrak{p}(z)}{\alpha^2}, \quad (26)$$

$$z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z) = \frac{z^4\mathfrak{p}^{(3)}(z)+3(\alpha-1)z^3\mathfrak{p}''(z)+(2\alpha^2+5\alpha+1)z^2\mathfrak{p}'(z)+\alpha^3 z\mathfrak{p}(z)}{\alpha^3}, \quad (27)$$

and

$$z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z) = \frac{z^5\mathfrak{p}^{(4)}(z)+4\alpha z^4\mathfrak{p}^{(3)}(z)+(5\alpha^2+8\alpha-5)z^3\mathfrak{p}''(z)}{\alpha^4} + \frac{(2\alpha^3+7\alpha^2+6\alpha+1)z^2\mathfrak{p}'(z)+\alpha^4 z\mathfrak{p}(z)}{\alpha^4}. \quad (28)$$

Define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by:

$$u(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \mathfrak{r}, \; v(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{s}+\alpha\mathfrak{r}}{\alpha},$$
$$x(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{t}+(2\alpha+1)\mathfrak{s}+\alpha^2\mathfrak{r}}{\alpha^2}, \quad (29)$$
$$y(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{w+3(\alpha-1)\mathfrak{t}+(2\alpha^2+5\alpha+1)\mathfrak{s}+\alpha^3\mathfrak{r}}{\alpha^3},$$

and

$$g(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b}) = \frac{\mathfrak{b}+4\alpha w+(5\alpha^2+8\alpha-5)\mathfrak{t}+(2\alpha^3+7\alpha^2+6\alpha+1)\mathfrak{s}+\alpha^4\mathfrak{r}}{\alpha^4}. \quad (30)$$

let

$$\chi(\mathfrak{r},\mathfrak{s},\mathfrak{t},w,\mathfrak{b};z) = \phi(u,v,x,y,g;z) = \phi\left(\begin{array}{c}\mathfrak{r}, \frac{\mathfrak{s}+\alpha\mathfrak{r}}{\alpha}, \\ \frac{\mathfrak{t}+(2\alpha+1)\mathfrak{s}+\alpha^2\mathfrak{r}}{\alpha^2}, \frac{w+3(\alpha-1)\mathfrak{t}+(2\alpha^2+5\alpha+1)\mathfrak{s}+\alpha^3\mathfrak{r}}{\alpha^3}, \\ \frac{\mathfrak{b}+4\alpha w+(5\alpha^2+8\alpha-5)\mathfrak{t}+(2\alpha^3+7\alpha^2+6\alpha+1)\mathfrak{s}+\alpha^4\mathfrak{r}}{\alpha^4}; z\end{array}\right). \quad (31)$$

Lemma (1) will be used in the proof. Using the Eqs. (24)-(28), we have from Eq. (31), that

$$\chi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z) = \phi\left(\begin{array}{c} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{array}\right). \quad (32)$$

Hence Eq. (23) becomes:

$$\chi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z) \in \Omega.$$

We see that

$$\frac{\mathfrak{t}}{s} + 1 = \frac{(x+v)+\alpha(3v+\alpha u}{\alpha(v+\alpha u)},$$
$$\frac{w}{\mathfrak{s}} = \frac{(y-3x+v)+\alpha(3x+5v)+\alpha^2(2v+\alpha u}{\alpha^2(v+\alpha u)},$$

and

$$\frac{\mathfrak{b}}{\mathfrak{s}} = \frac{(g+v-5x)+\alpha^2(5x+7v)+2\alpha(4x+3v)+\alpha^3(2v+\alpha u)}{\alpha^3(v+\alpha u)}.$$

As a result, it is evident that the admissibility requirement for $\phi \in M_{i,1}[\Omega,\mathfrak{q}]$ in Definition (7) is equal to the admissibility condition for $\chi \in \Psi_3[\Omega,\mathfrak{q}]$ provided in Definition (4) for $n = 3$. Consequently, applying Lemma (1) and Eq. (22) yields:

$$\mathfrak{p}(z) = z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z).$$

The proof of Theorem (4) is thus finished.

In our corollary we gain an extension of Theorem (4), to the scenario when the behavior of $\mathfrak{q}(z)$ on $\partial\mathbb{U}$ is not known.

Corollary (3): Let $\Omega \subset \mathbb{C}$ and let the function $\mathfrak{q}(z)$ be univalent in $\mathbb{U}$ with $\mathfrak{q}(o) = 1$. Let $\phi \in M_{i,1}[\Omega,\mathfrak{q}]$ for some $\rho(o,1)$ where $\mathfrak{q}_\rho(z) = \mathfrak{q}(\rho z)$. If $f \in D_u$ and $\mathfrak{q}_\rho$, satisfies the following conditions:

$$\mathcal{R}e\left(\frac{\zeta^2\mathfrak{q}'''(\zeta)}{\mathfrak{q}'(\zeta)}\right) \geqq 0, \quad \left|\frac{z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(\zeta)}\right| \leqq k^2, \quad (33)$$
$$\left(z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathfrak{q}_\rho) \text{ and } k \geqq n\right)$$

and

$$\phi\left(\begin{array}{c} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{array}\right) \prec \mathfrak{h}(z), \quad (34)$$

then

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z) \quad (z \in \mathbb{U}).$$

Proof: By using Theorem (4), yield

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}_\rho(z) \quad (z \in \mathbb{U}),$$

then we obtain the result from

$$\mathfrak{q}_\rho(z) \prec \mathfrak{q}(z) \quad (z \in \mathbb{U}).$$

This completes Corollary (3)'s proof.

$\Omega = \mathfrak{h}(\mathbb{U})$ for some conformal mapping $h(z)$ of U on to Ω if $\Omega \neq \mathbb{C}$ is a simply-connected domain. The class $M_{i,1}[\mathfrak{h}(\mathbb{U}), \mathfrak{q}]$, is expressed as $M_{i,1}[\mathrm{h},\mathfrak{q}]$ in this instance. Theorem (4) and Corollary (3) immediately lead to the next two conclusions.

Theorem (5): Let $\phi \in M_{i,1}[\Omega,,\mathfrak{q}]$ If $f \in D_u$ and $\mathfrak{q} \in \mathcal{Q}_1$, satisfy the conditions Eq. (22) and

$$\phi\left(\begin{array}{c} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{array}\right) \prec \mathfrak{h}(z), \quad (35)$$

then

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z) \quad (z \in \mathbb{U}).$$

Corollary (4): Given a function $q(z)$ that is univalent in U and $q(0) = 1$, *let* $\Omega \subset \mathbb{C}$. For any $\rho \in (0,1)$, such that $p_\rho(z) = p(\rho z)$, let $\phi \in M_{i,1}[\Omega,\mathfrak{q}]$. In the event that $f \in D_u$ and $\mathfrak{q}_\rho$, satisfy the requirements (22):

$$\phi\left(\begin{array}{c} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z)\end{array}\right) \prec \mathfrak{h}(z), \quad (36)$$

then

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z) \quad (z \in \mathbb{U}).$$

The best dominant of differential subordination is yielded in the following result by Eq. (35).

Theorem (6): Assume that in U, the function h is univalent. Furthermore, let $\phi: \mathbb{C}^5 \times \mathbb{U} \to \mathbb{C}$, and assume that the differential equation.

$$\phi\left(\begin{array}{c} q(z), \frac{z^2\mathfrak{q}'(z)+\alpha z\mathfrak{q}(z)}{\alpha}, \frac{z^3\mathfrak{q}''(z)+(2\alpha+1)z^2\mathfrak{q}'(z)+\alpha^2 z\mathfrak{q}(z)}{\alpha^2}, \\ \frac{z^4\mathfrak{q}^{(3)}(z)+3(\alpha-1)z^3\mathfrak{q}''(z)+(2\alpha^2+5\alpha+1)z^2\mathfrak{q}'(z)+\alpha^3 z\mathfrak{q}(z)}{\alpha^3}, \\ \frac{z^5\mathfrak{q}^{(4)}(z)+4\alpha z^4\mathfrak{q}^{(3)}(z)+(5\alpha^2+8\alpha-5)z^3\mathfrak{q}''(z)}{\alpha^4} + \\ \frac{(2\alpha^3+7\alpha^2+6\alpha+1)z^2\mathfrak{q}'(z)+\alpha^4 z\mathfrak{q}(z)}{\alpha^4}; z \end{array}\right) = \mathfrak{h}(z), \quad (37)$$

has a solution $\mathfrak{q}(z)$ with $\mathfrak{q}(0) = 1$, which satisfies the requirement (34). Should f be outside of D_u, then

$$z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z) \prec \mathfrak{q}(z), \qquad (z \in \mathbb{U}).$$

Proof: It follows that $z^{\mathfrak{p}}\mathcal{H}_{\mathfrak{p}}(\tau,\psi)(\mathfrak{f} * \mathfrak{g})(z) \prec \mathfrak{q}_\rho(z)$. from Theorem (4). The following subordination property, $\mathfrak{q}_\rho(z) \prec \mathfrak{q}(z)$ $(z \in \mathbb{U})$, leads to the conclusion stated in Corollary (1).

Additionally, the integral transformation $\mathcal{J}_{(\alpha,\beta)}f(z)$ defined in (2) is examined in this case using the fourth-order differential superordination thermos. For the purpose, we considered the following acceptable functions.

3. RESULTS OF FOURTH-ORDER DIFFERENTIAL SUPERORDINATION WITH $\mathcal{J}_{(\alpha,\beta)}f(z)$

Definition (8): Given $\mathfrak{q} \in \mathcal{Q}_0$ and $\mathfrak{q}'(z) \neq 0$, let Ω be a set in C. Included in the class $\mathrm{M}_i[\Omega, \mathfrak{q}]$ of admissible functions are those functions $\phi: \mathbb{C}^5 \times \overline{\mathbb{U}} \to \mathbb{C}$, that meet the admissibility requirement mentioned below.

$$\phi(u, v, x, y, g; \zeta) \in \Omega,$$

when

$$u = \mathfrak{q}(z)\,,\ v = \frac{k\xi\mathfrak{q}'(\xi)+(\alpha-1)\mathfrak{q}(\xi)}{\alpha},$$
$$\mathcal{R}e\left(\frac{x+\alpha(2v+(\alpha-1)u)}{\alpha(u+\alpha(v-u)}\right) \leqq \frac{1}{m}\mathcal{R}e\left\{\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}+1\right\},$$
$$\mathcal{R}e\left(\frac{y+\alpha(3x+u)+2\alpha^2 v-(v+u)}{\alpha^2(v-u)+\alpha^3 u}\right) \leqq \frac{1}{m^2}\mathcal{R}e\left\{\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right\},$$

and

$$\mathcal{R}e\left(\frac{g + 2\alpha(2y+3x) - (y+4x) + 5\alpha^2 x + (\alpha-1)^4((2\alpha^2-1)^2 v + u)}{\alpha^3(v+(\alpha-1)u)}\right) \leqq \frac{1}{m^3}\mathcal{R}e\left(\frac{\zeta^3\mathfrak{q}''''(\zeta)}{\mathfrak{q}'(\zeta)}\right),$$

in contrast, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathfrak{q})$ and $k \geqq 3$.

Theorem (7): Let $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$. If $f \in D_u$ and $\mathcal{J}_{(\alpha,\beta)}f(z) \in \mathcal{Q}_0$ with $q'(z) \neq 0$, satisfying the following conditions:

$$\mathcal{R}e\left(\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \left|\frac{\mathcal{J}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(z)}\right| \leqq \frac{1}{m^2}, \quad (38)$$

and the function $\phi(\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z)$, is univalent in $\mathbb{U}$, and

$$\Omega \subset \left\{\phi\left(\begin{array}{c}\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \\ \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z\end{array}\right): z \in \mathbb{U}\right\}, \quad (39)$$

suggest that

$$\mathfrak{q}(z) \prec \mathcal{J}_{(\alpha,\beta)}f(z) \qquad (z \in \mathbb{U}).$$

Proof: Let the function $\mathfrak{p}(z)$ be defined by Eq. (7) and χ by (14), since $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$. Thus from Eqs. (15) and (39) yield

$$\Omega \subset \left\{\chi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z\right): z \in \mathbb{U}\right\}.$$

We can observe from Eqs. (12) and (13), that the admissibility condition forχ as statedin Definition (5), with $n = 3$, is identical to the admissibility for ϕ in Definition (6) for $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$. Since $\chi \in \Psi_3[\Omega, \mathfrak{q}]$, we can use Lemma (2) and (39) to obtain

$$\mathfrak{q}(z) \prec \mathcal{J}_{(\alpha,\beta)}f(z).$$

$\Omega = \mathfrak{h}(\mathbb{U})$ for some conformal mapping $h(z)$ of U on to Ω if $\Omega \neq \mathbb{C}$ is a simply-connected domain. In this instance, a $\mathrm{M}_i[\mathfrak{h}\,, \mathfrak{q}]$, is written for the class $\mathrm{M}_i[\mathfrak{h}(\mathbb{U})\,, \mathfrak{q}]$.

Theorem (7) naturally leads to the next theorem.

Theorem (8): Let $\phi \in \mathrm{M}_i[\Omega, \mathfrak{q}]$ and $\mathfrak{h}$ be analytic functioning U If $f \in D_u, \mathcal{J}_{(\alpha,\beta)}f(z) \in \mathcal{Q}_0$ and $\mathfrak{q} \in \mathcal{J}_0$ with $\mathfrak{q}'(z) \neq 0$, satisfying the following conditions (38) and

$$\phi(\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z),$$

is univalent in $\mathbb{U}$, then

$$\mathfrak{h}(z) \prec \phi\left(\begin{array}{c}\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \\ \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z\end{array}\right), \quad (40)$$

suggest that

$$\mathfrak{q}(z) \prec \mathcal{J}_{(\alpha,\beta)}f(z) \qquad (z \in \mathbb{U}).$$

Theorem (9): Let $\mathfrak{h}$ be analytic function in $\mathbb{U}$, and let $\phi: \mathbb{C}^5 \times \overline{\mathbb{U}} \to \mathbb{C}$ and χ be given by Eq. (14). Suppose that the following differential equation:

$$\chi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z\right) = \mathfrak{h}(z), \quad (41)$$

has a solution $\mathfrak{q}(z) \in \mathcal{Q}_0$. If $f \in D_u, \mathcal{J}_{(\alpha,\beta)}f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{J}_0$ with $q'(z) \neq 0$, satisfying the following conditions (38) and $\phi(\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z)$, is univalent in $\mathbb{U}$, then

$$\mathfrak{h}(z) \prec \phi\left(\begin{array}{c}\mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \\ \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z\end{array}\right),$$

suggests that

$$\mathfrak{q}(z) \prec \mathcal{J}_{(\alpha,\beta)} f(z) \qquad (z \in \mathbb{U}),$$

and $q(z)$ is the best subordinant of (40).

Proof: We observe that q is a subordination of (40) in light of Theorems (7) and (8). All subordinants will subordinate q since it fulfills (41), which also makes it a solution of (40). Thus, the best subordinant is q.

Definition (10): Given a set Ω in $\mathbb{C}$ and $q \in \mathcal{Q}_1$ with $q'(z) \neq 0$. Let any function $\phi: \mathbb{C}^5 \times \overline{\mathbb{U}} \to \mathbb{C}, that$ meet the following admissibility requirement are included in the class $\mathrm{M}_{i,1}[\Omega, \mathfrak{q}]$ of admissible functions.

$$\phi(u, v, x, y, g; \zeta) \in \Omega,$$

when

$$u = \mathfrak{q}(z),$$
$$v = \frac{k\xi \mathfrak{q}'(\xi) + \alpha \mathfrak{q}(\xi)}{\alpha},$$
$$\mathcal{R}e\left(\frac{(x+v)+\alpha(3v+\alpha u)}{\alpha(v+\alpha u)}\right) \geqq \frac{1}{m}\mathcal{R}e\left\{\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right\},$$
$$\mathcal{R}e\left(\frac{(y-3x+v)+\alpha(3x+5v)+\alpha^2(2v+\alpha u)}{\alpha^2(v+\alpha u)}\right) \geqq \frac{1}{m^2}\mathcal{R}e\left\{\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right\}$$

and

$$\mathcal{R}e\left(\frac{(g+v-5x)+\alpha^2(5x+7v)+2\alpha(4x+3v)+\alpha^3(2v+\alpha u)}{\alpha^3(v+\alpha u)}\right) \geqq \frac{1}{m^3}\mathcal{R}e\left(\frac{z^3\mathfrak{q}''''(z)}{\mathfrak{q}'(z)}\right),$$

When, $z \in \mathbb{U}, \zeta \in \partial\mathbb{U} \setminus \mathcal{E}(\mathfrak{q})$ and $k \geqq 3$.

Theorem (10): Let $\phi \in \mathrm{M}_{i,1}[\Omega, \mathfrak{q}]$. If $f \in D_u$ and $z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z) \in \mathcal{Q}_1$ with $\mathfrak{q}'(z) \neq 0$, meeting the prerequisites listed below:

$$\mathcal{R}e\left(\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \left|\frac{z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z)}{\mathfrak{q}'(z)}\right| \leqq \frac{1}{m^2}, \qquad (42)$$

and the function $\phi\begin{pmatrix} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z) \\ , z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix}$, is univalent in $\mathbb{U}$, and

$$\Omega \subset \left\{\phi\begin{pmatrix} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix} : z \in \mathbb{U}\right\}, \qquad (43)$$

implies that

$$\mathfrak{q}(z) \prec z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z) \qquad (z \in \mathbb{U}).$$

Proof: Let the function $\mathfrak{p}(z)$ be defined by (24) and χ by (31), since $\phi \in \mathrm{M}_{i,1}[\Omega, \mathfrak{q}]$. Thus from (32) and (43) yield

$$\Omega \subset \{\chi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}^{(3)}(z), z^4\mathfrak{p}^{(4)}(z); z) : z \in \mathbb{U}\}.$$

The admissibility for $\phi \in \mathrm{M}_{i,1}[\Omega, \mathfrak{q}]$, as stated in Definition (6), is comparable to the admissibility requirement forχ as stated in Definition (5), with n=3, as can be seen from Eqs. (29) and (30). Since $\chi \in \Psi_3[\Omega, \mathfrak{q}]$, we can use Lemma (2) and (43) to obtain

$$\mathfrak{q}(z) \prec z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z).$$

For any conformal mapping $\mathfrak{h}(z)$ of $\mathbb{U}$ on to Ω, $\Omega = h(\mathbb{U})$ if $\Omega \neq \mathbb{C}$ is a simply connected domain. Here, the class $\mathrm{M}_{i,1}[\mathfrak{h}, \mathfrak{q}]$. $\mathrm{M}_{i,1}[\mathfrak{h}(\mathbb{U}), \mathfrak{q}]$ is expressed as a $\mathrm{M}_{i,1}[h, q]$. This theorem simply follows from the previous one (10).

Theorem (11): Let $\phi \in \mathrm{M}_{i,1}[\Omega, \mathfrak{q}]$ and h be analytic function in $\mathbb{U}$. If $f \in D_u$ and $\mathfrak{q} \in \mathcal{Q}_1$ with $q'(z) \neq 0$, satisfying the following conditions (42) and:

$$\phi\begin{pmatrix} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix},$$

is univalent in $\mathbb{U}$, then

$$\mathfrak{h}(z) \prec \phi\begin{pmatrix} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z), \\ z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix}, \qquad (44)$$

implies that

$$\mathfrak{q}(z) \prec z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z) \qquad (z \in \mathbb{U}).$$

4. SANDWICH EFFECTS

Combining Theorems (2) and (8), we obtain the following sandwich-type theorem.

Theorem (12): Let h_1 and q_1 ,be univalent in $\mathbb{U}$, h be univalent function in $\mathbb{U}$, $\mathfrak{q}_2 \in \mathcal{Q}_0$, with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0) = 1$ and $\phi \in \mathrm{M}_i[\mathfrak{h}_1, \mathfrak{q}_1] \cap \mathrm{M}_i[\mathfrak{h}_2, \mathfrak{q}_2]$.

If, $f \in D_u$, $\mathcal{J}_{(\alpha,\beta)} f(z) \in \mathcal{Q}_0 \cap \mathcal{J}_0$, and $\phi\begin{pmatrix} \mathcal{J}_{(\alpha,\beta)}f(z), \mathcal{J}_{(\alpha+1,\beta)}f(z), \mathcal{J}_{(\alpha+2,\beta)}f(z), \\ \mathcal{J}_{(\alpha+3,\beta)}f(z), \mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix}$, is univalent in U, and the fulfillment of requirements (5) and (38) occurs, then

$$\mathfrak{q}_1(z) \prec \mathcal{J}_{(\alpha,\beta)} f(z) \prec \mathfrak{q}_2(z).$$

Combining Theorems (6) and (11), we obtain the following sandwich-type theorem.

Theorem (13): Let h_1 and q_1 , be univalent in $\mathbb{U}$, h be univalent function in $\mathbb{U}$, $\mathfrak{q}_2 \in \mathcal{Q}_1$, with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0) = 1$ and $\phi \in \mathrm{M}_{i,1}[\mathfrak{h}_1, \mathfrak{q}_1] \cap \mathrm{M}_{i,1}[\mathfrak{h}_2, \mathfrak{q}_2]$. If $f \in D_u$, $z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z) \in \mathcal{Q}_1 \cap \mathcal{J}_1$, and

$$\phi\begin{pmatrix} z^{-1}\mathcal{J}_{(\alpha,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+1,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+2,\beta)}f(z) \\ , z^{-1}\mathcal{J}_{(\alpha+3,\beta)}f(z), z^{-1}\mathcal{J}_{(\alpha+4,\beta)}f(z); z \end{pmatrix},$$

fulfills the requirements of Eqs. (22) and (42) and is univalent in U, then

$$\mathfrak{q}_1(z) \prec z^{-1}\mathcal{J}_{(\alpha,\beta)} f(z) \prec \mathfrak{q}_2(z).$$

5. DISCUSSION AND CONCLUSION

Using an integral operator $\mathcal{J}_{(\alpha,\beta)}$, of analytic functions in U, we study appropriate classes of admissible functions and establish the properties of fourth-order differential subordination and superordination. We have the normalized Taylor-Maclaurin series of the following form: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $(z \in U)$. We establish some novel conclusions on superordination and differential subordination with a few corollaries. Additionally, we get multiple sandwich-style outcomes. Our findings diverge from the other authors' earlier findings. Using the findings in the paper, we provided several opportunities for writers to expand our new subclasses and produce new findings in univalent and multivalent function theory, it can be done to study ideal classes of admissible functions to determine the features of fourth-order differential subordination and superordination by making apply fresh conditions of analytic functions and got fresh sandwich results or by with the same classes of admissible functions in this work using a different operator.

REFERENCES

[1] Duren, P.L. (1983). Univalent functions. In: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Hidelberg and Tokyo.

[2] Darweesh A.M. (2022). Some Studies on Differential Subordination and Quasi –Subordination of Subclasses in Geometric Function Theory, University of Kufa, Najaf, Iraq.

[3] AL-Ameedee, S.A., Atshan, W.G., Ali AL-Maamori, F. (2020). On sandwich results of univalent functions defined by a linear operator. Journal of Interdisciplinary Mathematics, 23(4): 803-809. https://doi.org/10.1080/09720502.2020.1727613

[4] Ali, R.M., Ravichandran, V., Seenivasagan, N. (2009). Differential subordination and superordination of analytic functions defined by the multiplier transformation. Mathematical Inequalities & Applications, 12(1): 123-139. https://doi.org/10.7153/mia-12-11

[5] Cho, N.E., Yoon, M. (2011). Subordination properties for meromorphic multivalent functions associated with the multiplier transformation. Applied Mathematics and Computation, 218(3): 729-733. https://doi.org/10.1016/j.amc.2011.02.008

[6] Cotîrlă, L. (2009). A differential sandwich theorem for analytic functions defined by the integral operator. Studia Universitatis Babes-Bolyai, Mathematica, 54(2): 13-21.

[7] Miller, S.S., Mocanu, P.T. (2000). Differential Subordinations: Theory and Applications. In: Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, Incorporated, New York and Basel.

[8] Murugusundaramoorthy, G., Magesh, N. (2008). An application of second order differential inequalities based on linear and integral operators. International Journal of Mathematical Sciences and Engineering Applications (IJMSEA), 2(1): 105-114.

[9] Sarangi, S.M., Uralegaddi, S.B. (1996). Certain differential operators for meromorphic functions. Bulletin of the Australian Mathematical Society, 88: 333-336.

[10] Aouf, M.K., Seoudy, T.M. (2010). Subordination and superordination of a certain integral operator on meromorphic functions. Computers & Mathematics with Applications, 59(12): 3669-3678. https://doi.org/10.1016/j.camwa.2010.03.068

[11] Atshan, W.G., Hadi, R.A. (2020). Some differential subordination and superordination results of p-valent functions defined by differential operator. Journal of Physics: Conference Series, 1664(1): 012043. IOP Publishing. https://doi.org/10.1088/1742-6596/1664/1/012043

[12] Miller, S.S., Mocanu, P.T. (2003). Subordinants of differential superordinations. Complex Variables, 48(10): 815-826. https://doi.org/10.1080/02781070310001599322

[13] Cho, N.E., Bulboacă, T., Srivastava, H.M. (2012). A general family of integral and associated subordination and superordination properties of analytic function classes. Applied Mathematics and Computation, 219(4): 2278-2288. http://dx.doi.org/10.1016/j.amc.2012.08.075

[14] Ponnusamy, S., Juneja, O.P. (1992). Third-order differential inqualities in the complex plane. In Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.

[15] Uralegaddi, B.A., Somanatha, C. (1991). New criteria for meromorphic starlike univalent functions. Bulletin of the Australian Mathematical Society, 43(1): 137-140. https://doi.org/10.1017/S0004972700028859

[16] Antonino, J.A., Miller, S.S. (2011). Third-order differential inequalities and subordinations in the complex plane. Complex Variables and Elliptic Equations, 56(5): 439-454. https://doi.org/10.1080/17476931003728404

[17] Tang, H., Srivastava, H.M., Li, S.H., Ma, L.N. (2014). Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator. Abstract and Applied Analysis, 2014: 792175. https://doi.org/10.1155/2014/792175

[18] Atshan, W.G., Battor, A.H., Abaas, A.F. (2020). On third-order differential subordination results for univalent analytic functions involving an operator. Journal of Physics: Conference Series, 1664(1): 012041. https://doi.org/10.1088/1742-6596/1664/1/012041

[19] Darweesh, A.M., Atshan, W.G., Battor, A.H., Lupaş, A.A. (2022). Third-order differential subordination results for analytic functions associated with a certain differential operator. Symmetry, 14(1): 99. https://doi.org/10.3390/sym14010099

[20] Atshan, W.G., Battor, A.H., Abaas, A.F., Oros, G.I. (2020). New and extended results on fourth-order differential subordination for univalent analytic functions. Al-Qadisiyah Journal of Pure Science, 25(2): 1-13. https://doi.org/10.29350/jops.2020.25.2.1066

[21] Atshan, W.G., Abbas, I.A., Yalcin, S. (2020). New concept on fourth-order differential subordination and superordination with some results for multivalent analytic functions. Journal of Al-Qadisiyah for Computer Science and Mathematics, 12(1): 96-107. https://doi.org/10.29304/jqcm.2020.12.1.681