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# Dominance Parameters in Prism Graphs: A Comparative Study of Minimum Dominating Sets

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**ABSTRACT**

Let $G=(V, E)$ be a graph. A dominating set $S$ of graph $G$ is defined as a set of vertices such that every vertex in $V\setminus S$ is adjacent to at least one vertex in $S$. The domination number of graph $G$, denoted as $\gamma(G)$, corresponds to the size of the smallest dominating set within $G$. In other words, $\gamma(G)$ represents the number of vertices required in the minimum dominating set to cover all other vertices in the graph $G$. In the graph $G$, our objective is to position a protector at each vertex within a subset $S$ of $V$, ensuring that $S$ forms a dominating set, effectively covering all other vertices in $G$. Moreover, in the event that a protector positioned at vertex $v$ needs to move along an edge to protect an unguarded vertex $u$, the arrangement of protectors should maintain the property of forming a dominating set for the graph. In other words, the movement of protectors should maintain the property of domination within the graph, ensuring efficient coverage and defense across the network. The bare minimum of security guards is necessary to protect all vertices in the graphs. In this article, we find the bounds for domination, independent domination number (IDN), connected domination number(CDN), total domination number(TDN), and the secure domination number(SDN) denoted by$\gamma(A_n)$, $\gamma_i(A_n)$, $\gamma_c(A_n)$, $\gamma_t$ and $\gamma_s(A_n)$ respectively for the antiprism graph, where $A_n$ denoted the 4 - regular graph with girth 3. We further establish that the TDN is greater than or equal to the SDN of the antiprism graph for $n \geq 3$.

## 1. INTRODUCTION

In the realm of graph theory, the properties and characteristics of graphs, particularly those without loops or multiple edges, hold significant intrigue. Such graphs, denoted as $G = (V, E)$, are characterized as finite and undirected. Within this context, a graph $G$ has ascribed the label of a $d$-regular graph [1, 2]. When each vertex in the set $V(G)$ displays a precisely uniform degree of $d$.

Pioneering explorations into the intricacies of regular graphs have brought forth a pertinent concept known as the girth, encapsulating the shortest cycle length discernible within $G$ [3]. The exploration of vertices takes us to the notion of open and closed neighborhoods. Specifically, the open neighborhood of a vertex $v \in V$ is aptly defined as $N(v) = \{u \in V: uv \in E\}$. An extension of this concept, the closed neighbourhood $N[v] = N(v) \cup \{v\}$, comprises the union of $N(v)$ and the vertex $v$ itself. This framework lays the foundation for the definition of private neighbours. A vertex $u \in V$is an S-private neighbour of $v$, where $S \subseteq V$ and $v \in S$, if the intersection of their neighbourhoods is $\{v\}$, i.e., $N[u] \cap S = \{v\}$. The set of all $S$-private neighbours of $v$, its denoted by $PN(v, s)$. Further nuances emerge when considering the case of $u \in V\setminus S$, designating $u$ as an $S-$external private neighbour of $v$.

In the realm of domination theory, a pivotal concept surfaces-dominating set. A dominating set $S$ within a graph $G$ is a congregation of vertices where each vertex not in $S$ has a direct connection to at least one vertex within $S$. The cardinality of the smallest dominating set in $G$ is encapsulated in the domination number $\gamma(G)$. In a graph $G$, an independent dominating set refers to a dominating set $S$ in which no two vertices are adjacent to each other. The IDN of graph $G$ is denoted by $\gamma_i(G)$, and represents the size of the smallest independent dominating set in $G$ [4, 5].

Broadening our perspective on connected domination, a set $S$ of vertices constitutes a connected dominating set if it satisfies two essential conditions. First, $S$ serves as a dominating set in $G$. Second, the subgraph engendered by the vertices of S is intrinsically connected. The CDN of graph $G$, denoted by $\gamma_c(G)$, represents the size of the smallest connected dominating set in $G$ [6, 7].

A total dominating set of $G$ is defined as a dominating set $S$ in which the induced subgraph $\langle S\rangle$ has no isolated vertex [8, 9]. It is implied that every vertex in $S$ has atleast one adjacent vertex in $S$, which means that all the vertex in $S$ are connected.

The TDN of graph G, denoted by $\gamma_t(G)$, represents the size of the smallest total dominating set in $G$.

A secure dominating set in a graph $G$ refers to a dominating set S that satisfies two conditions: First, for every vertex $u$ in the set $V\setminus S$, there exists a vertex $v$ in $S$ such that $u$ and $v$ are adjacent to each other. Second, if we remove vertex $v$ from $S$ and add vertex $u$, forming the set $S_1 = (S\setminus\{v\}) \cup \{u\}$, $S_1$ also becomes a dominating set. The SDN of graph $G$ represents the size of the smallest secure dominating set in $G$, and it is denoted by $\gamma_s(G)$. Cockayne et al. were the first to introduce this concept. Several authors have studied it further [10-12].

The thought of secure domination finds application in specific scenarios wherein the vertex set of $G$ defines distributed locations within a spatial domain, and the edges of $G$ symbolize occupied connections between these locations. Patrolling guards move along the links to safeguard the graph, ensuring the protection of both the connections and the individual positions at each of these sites.

The smallest essential assembly of positions in $G$, known as a minimum secure dominating set encompasses locations strategically occupied by guards. This arrangement ensures the overall safety and assurance of the entire site complex depicted by $G$. In this scheme, if a security concern arises at a site, it can be addressed either by a guard stationed at that specific location or by a guard situated in an adjacent position. Regardless of the resolution approach taken, the site's security remains intact even after the guard involved has relocated. When considering the aforementioned applications, the concept of edge domination becomes significant due to its ability to provide threshold insights into the cumulative failures of edges. These insights, in turn, prove valuable for making informed decisions about increasing the number of guards to address vulnerabilities within the location complex.

Cayley graphs serve as effective models for interconnection networks, primarily because they are vertex-transitive graphs. This relevance is underscored by the fact that the majority of modern computers rely on large-scale parallel computing and inherently feature these interconnection networks.

Our primary concern revolves around graphs in which the domination number of some or all of the prisms is equivalent to double the domination number of the graph itself. In this article, we find an IDN, CDN, total domination, and the secure domination of antiprism graph $A_n$. We further established that $\gamma(A_n) = \gamma_i(A_n)$ for $n \geq 3$, $\gamma_c(A_n) = n-1$ for $n \geq 3$, and also obtained the TDN and SDN of the antiprism graph.

## 2. PRELIMINARIES

In this section, we discuss the construction of antiprism graphs and their domination properties. The definitions and the associated theorems needed for further sections are incorporated. In this paper, the prism graph is denoted by $D_n$, and the antiprism graph is denoted by $A_n$.

### 2.1 The quartic graph $Q_n$

"The structure of the Quartic graph with girth 4 is defined as follows. Thus $v_1$ is adjacent with $v_{n-1}, v_n, v_2, v_4$; $v_2$ is adjacent with $v_{n-1}, v_1, v_3, v_5$; $v_3$ is adjacent with $v_n, v_2, v_4, v_6$; $v_i$ is adjacent with $v_{i-1}, v_{i-3}, v_{i+1}, v_{i+3}$, where $i = 4$ to $n-3$, $v_{n-2}$ is adjacent with $v_{n-5}, v_{n-3}, v_{n-1}, v_1$; $v_{n-1}$ is adjacent with $v_{n-4}, v_{n-2}, v_n, v_2$ and $v_n$ is adjacent with $v_{i-1}, v_{i-3}, v_1, v_3$. Each vertex is obviously of degree 4 and it is given in Figure 1. Thus, the graph has edges of $2n$. So, we have the Quartic graph of order $n$ with edges of girth 4 and $2n$ from the structure."

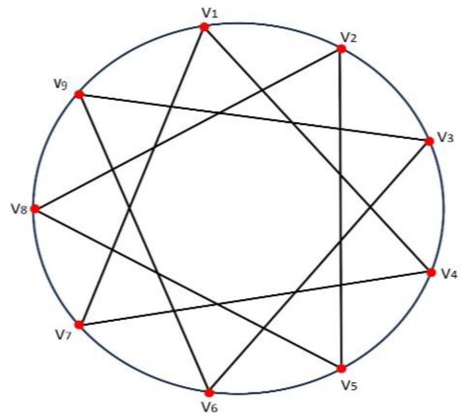



**Figure 1.** Quartic Graph $Q_8$

2.1.1 Dominating set of $Q_n$

For $n > 9$, a subset $S \subset V(Q_n)$ is a dominant set with girth 4 of an $n$ −vertex of Quartic graph $Q_n$. $S$ is defined by:

$$S = \begin{cases} A \text{ if } n \equiv 0,1,2,3 \pmod 5 \\ A \cup \{v_{n-1}\} \text{ if } 4 \pmod 5 \end{cases}$$

2.1.2 Independent dominating set of $Q_n$

For $n > 9$, a subset $S \subset V(Q_n)$ is an independent dominating set with girth 4 of a $n$ −vertex of Quartic graph $Q_n$. $S$ is defined by:

$$S = \begin{cases} A \text{ if } n \equiv 0,2 \pmod 5 \\ B \cup \{v_{n-1}\} \text{ if } n \equiv 4 \pmod 5 \\ C \cup \{v_{n-3}\} \cup \{v_{n-1}\} \text{ if } n \equiv 1 \pmod 5 \\ D\{v_{n-5}\} \cup \{v_{n-3}\} \cup \{v_{n-1}\} \text{ if } n \equiv 3 \pmod 5 \end{cases}$$

where,

$$A = \{v_{5t+1}: t \geq 0, \text{if } 5t+1 \leq n\}$$
$$B = \{v_{5t+1}: t \geq 0, \text{if } 5t+1 < n-1\}$$
$$C = \{v_{5t+1}: t \geq 0, \text{if } 5t+1 < n-3\}$$
$$D = \{v_{5t+1}: t \geq 0, \text{if } 5t+1 < n-5\}$$

2.1.3 Connected dominating set of $Q_n$

For $n > 9$, a subset $S \subset V(Q_n)$ is connected the dominating set of an n-vertex Quartic graph with girth 4. $S$ is defined by:

$$S = \begin{cases} \{A = v_i: i = 1,4,7,10,\ldots,n-4\}, for\ n = 8,11,14,\ldots \\ \{B = v_i: i = 1,4,7,10,\ldots,n\}, \quad for\ n = 10,13,16,\ldots \\ \{C = v_i: i = 1,4,7,10,\ldots,n-2\}, for\ n = 9,12,18,\ldots \end{cases}$$

### 2.2 The prism graph $D_n$

The Cartesian product of two graphs, denoted as $G_1 \times G_2$, is a new graph formed by combining every vertex of $G_1$ with every vertex of $G_2$, creating a vertex set where each vertex represents an ordered pair $(u, v)$, where $u$ is a vertex from is $G_1$ and $v$ is a vertex from $G_2$. Additionally, two of these ordered pairs are connected by an edge in the Cartesian product if and only if their corresponding vertices in $G_1$ and

$G_2$ are connected by edges in their respective graphs and it is mentioned in Figure 2. In essence, the Cartesian product generates a graph where the connectivity of each ordered pair is determined by the individual edge relationships in the original graphs $G_1$ and $G_2$.

The prism graph $D_{\mathrm{n}}$ can be constructed by taking the Cartesian product of the cycle $C_{\mathrm{n}}$ and the path $P_2$ [13], resulting in a 3-regular graph depicted in Figure 3. According to the research conducted by Raza et al. [14] and Kartelj et al. [15], this graph belongs to the category of Archimedean convex polytopes. Furthermore, it should be emphasized that the prism graph can be considered identical to the Petersen graph, denoted as $P(n, 1)$. From a mathematical perspective, the set of vertices and edges in the prism graph $D_{\mathrm{n}}$ is represented as $V(D_{\mathrm{n}})$ and $E(D_{\mathrm{n}})$, respectively.

$$V(\mathrm{D}_n) = \{(v_k,\ u_k), \quad k = 1,2, \ldots., n\}$$
$$E(\mathrm{D}_n) = \{(v_k, \quad u_k), (v_k,\ v_{k+1}), (u_k, \quad u_{k+1}), \ k = 1,2, \ldots., n-1\} \cup (v_n,\ u_n) \cup (v_n,\ v_1) \cup (u_n,\ u_1).$$

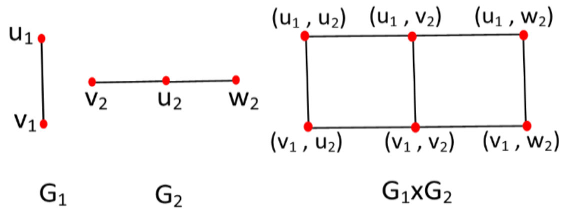



**Figure 2.** Cartesian product of Graph

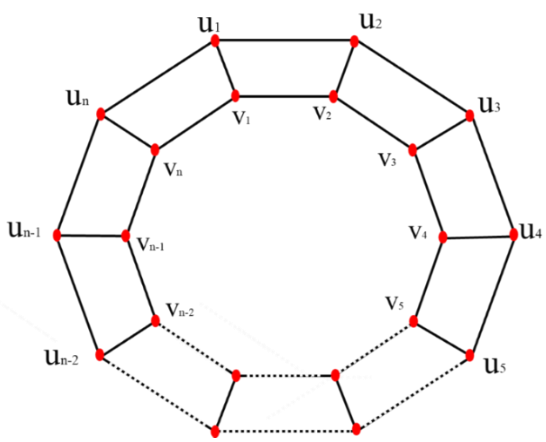



**Figure 3.** (a) Prism Graph $D_n$

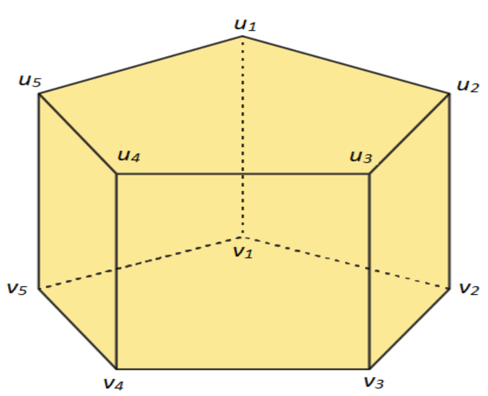



**Figure 3.** (b) Prism Graph $D_5$ with the 3D effect

**2.3 The antiprism graph $\mathrm{A_n}$**

The antiprism graph $A_n$ can be described as a 4-regular graph consisting of n - faces, 2n vertices, and 4n edges with girth 3, which is represented in Figure 4. The vertex set $V(A_n)$ and edge set $E(A_n)$ represent the sets of vertices and edges, respectively, in the antiprism graph $A_n$ [16]. The vertices on the inner cycle are denoted by $\{v_1, v_2, \ldots, v_{n,}\}$ and the vertices on the outer cycle are represented by $\{u_1, u_2, \ldots, u_n\}$. The inner vertex of the inner cycle is connected to the outer vertex of the outer cycle by two adjacent vertices. This is defined as follows:

$$V(A_n) = \{v_i \cup\ u_i, for\ 1 \leq \mathrm{i} \leq \mathrm{n}\}$$
$$\mathrm{E(A_n)} = \{(v_i,\ v_{i+1}) \cup (u_i,\ u_{i+1}) \cup (u_i,\ v_i) \cup (u_i,\ v_{i+1}), \ 1 \leq \mathrm{i} \leq \mathrm{n}-1\} \cup (u_n, u_1) \cup (v_n,\ v_1) \cup (v_n, u_n) \cup (v_1,\ u_n).$$

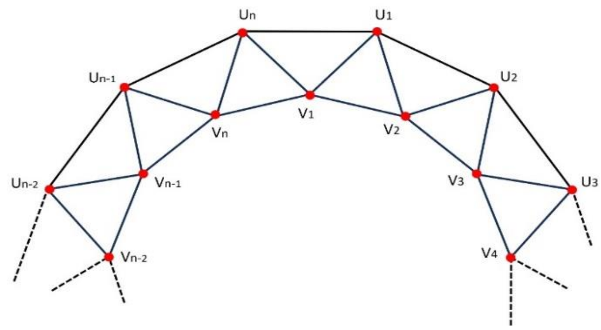



**Figure 4.** Antiprism Graph $\mathrm{A_n}$

2.3.1 Independent dominating set of $A_n$

Let's consider an independent dominating set $S$, where $S$ is a subset of the vertex set $V(A_n)$ of the antiprism graph $A_n$ for $\mathrm{n} \geq 3$. This set $S$ can be defined in the following way:

$$S = \begin{cases} A \cup \{u_{n-1}\} if\ n = 10r+1, 10r+3, 10r+6, 10r+8; \ r \geq 0 \\ A\ if\ n = 10r+4, 5r, 10r+7, 10r+9, 10r+2; \ r \geq 0 \end{cases}$$

where,

$$A = \{(v_i,\ \mathrm{i} = 1, 6, 11, 16, 21, \ldots \ldots., 5\mathrm{t}+1) \cup (u_i,\ \mathrm{i} = 3, 8, 13, 18, 23, \ldots \ldots.., 5\mathrm{t}+3),\ \mathrm{t} \geq 0, \mathrm{if}\ 5\mathrm{t}+1,\ 5\mathrm{t}+3 \leq \mathrm{n}-1\}.$$

2.3.2 Connected dominating set of $A_n$

Let's consider a connected dominating set $S$, where $S$ is a subset of the vertex set $V(A_n)$ of the antiprism graph $A_n$ for $\mathrm{n} \geq 3$. This set $S$ can be defined in the following way:

$$S = \{v_i\ ,\ \mathrm{i} = 1, 2, 3, 4, 5, \ldots \ldots \ldots, \mathrm{n}-1 \text{ for } \mathrm{n} \geq 3\}.$$

2.3.3 Total dominating set of $A_n$

Let's consider a total dominating set $S$, where $S$ is a subset of the vertex set $V(A_n)$ of the antiprism graph $A_n$ for $\mathrm{n} \geq 3$. This set S can be defined in the following way:

$$S = \{(v_i,\ \mathrm{i} = 1, 2, 8, 9, 15, \ldots., 7\mathrm{t}+1,\ 7\mathrm{t}+2) \cup (u_i,\ \mathrm{i} = 4, 5, 11, 12, 18, \ldots, 7\mathrm{t}+4,\ 7\mathrm{t}+5),\ \mathrm{t} \geq 0,\ \mathrm{if}\ 7\mathrm{t}+1,\ 7\mathrm{t}+2,\ 7\mathrm{t}+4,\ 7\mathrm{t}+5 \leq \mathrm{n} \text{ for } \mathrm{n} \geq 3\}.$$

**Theorem: 2.1** [17] For $n \geq 3$, let $G$ be a connected graph with n vertices, then $\gamma_t(G) \leq \frac{2n}{3}$.

**Theorem: 2.2** Let $G$ be a graph with $n$ vertices and minimum degree ($\delta(G)$) of at least 3, then $\gamma_t(G) \leq \frac{n}{2}$.

**Theorem: 2.3** [18] Let $C_n$ be the cycle with n- vertices, then $\gamma_s(C_n) = \left\lceil \frac{3n}{7} \right\rceil$ for all $n \geq 3$.

**Theorem: 2.4** [19] For the Quartic graph $Q_n (n > 9)$, the domination number,

$$\gamma(Q_n) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil & if\ n \equiv 0,1,2,3 \pmod 5 \\ \left\lceil \frac{n}{5} \right\rceil + 1 & if\ n \equiv 4 \pmod 5 \end{cases}$$

**Theorem: 2.5** [19] For the Quartic graph $Q_n (n > 9)$, the independent domination number,

$$\gamma_i(Q_n) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil & if\ n \equiv 0,2 \pmod 5 \\ \left\lceil \frac{n}{5} \right\rceil + 1 & if\ n \equiv 1,4 \pmod 5 \\ \left\lceil \frac{n}{5} \right\rceil + 2 & if\ n \equiv 3 \pmod 5 \end{cases}$$

**Theorem: 2.6** [19] For the Quartic graph $Q_n$ $(n > 9)$, the connected domination number,

$$\gamma_c(Q_n) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & if\ n \equiv 2 \pmod 3 \\ \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 0,1 \pmod 3 \end{cases}$$

## 3. RESULTS

In this segment, we analyse the parameter called the IDN, CDN, TDN, and SDN of the antiprism graph. These results indicate a relationship between the independent Domination Number IDN and other variables, such as the total domination number. Additionally, these findings establish upper bounds on the IDN that are optimal and depend on the graph's order.

**Theorem: 3.1** For the antiprism graph $A_n$ the independent domination number, $\gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for all $n \geq 3$.

**Proof:**

The independent dominating set $S$ of $A_n$ is Figure 4, it is clearly depicted that the vertex $v_1$ on the inner cycle dominates $\{v_n, u_1, u_n, v_2\}$. The vertex $v_3$ may be chosen for the dominating set as $v_2$ is already dominated by $v_1$. To obtain the independent dominating set, let us move to the vertex $u_3$ which dominates the vertices of the set $\{v_3, v_4, u_2, u_4\}$. In the same way, were attempting to find $\gamma_i(A_n)$ is minimum independent dominating set. The following details are:

**Case 1.** If $n = 10r + 1, 10r + 3, 10r + 6, 10r + 8;\ r \geq 0$, then $S = A \cup \{u_{n-1}\}$, where $A = \{\{v_{5t+1}, u_{5t+3};\ t \geq 0\}, if\ 5t + 1, 5t + 3 \leq n - 1\}$. From definition 2.3.1, every vertex in the set $S$ is non-adjacent to any other vertex within $S$. In such cases, the cardinality of set $S$, denoted as $|S|$, is equal to the number of vertices in $S$, which can be calculated as $\left\lceil \frac{2n}{5} \right\rceil$ for all $n \geq 3$. The reasoning for case $n = 6$ is presented, where the set of vertices $S = \{v_1, u_3, u_5\}$ is an IDN of $A_6$ and $|S| = 3 = \left\lceil \frac{2n}{5} \right\rceil$ .Similarly argument for the cases $n = 10r + 1, 10r + 3, 10r + 6, 10r + 8; r \geq 0$. On the contrary, if the set of vertices $S = \{v_1, u_2, v_4, u_5\}$ is an independent dominating set of $A_6$ & $|S| = 4$, then it is contradicting the definition of IDN [19]. Therefore, $|S| = 3 = \gamma_i(A_6) = \left\lceil \frac{2n}{5} \right\rceil$. Hence, $\gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for $n \geq 3$.

**Case 2.** If $n = 10r + 4, 5r, 10r + 7, 10r + 9, 10r + 2;\ r \geq 0$, then $S = \{A\}$ where $A = \{\{v_{5t+1}, u_{5t+3}; t \geq 0\}, if\ 5t + 1, 5t + 3 \leq n - 1\}$. From definition 2.3.1, every vertex in the set $S$ is non-adjacent to any other vertex within $S$. In such cases, the cardinality of set $S$, denoted as $|S|$, is equal to the number of vertices in $S$, which can be calculated as $\left\lceil \frac{2n}{5} \right\rceil$ for all $n \geq 3$ . The reasoning for the case $n = 5$ is presented, where the set of vertices $S = \{v_1, u_3\}$ is an IDN of $A_5$ and $|S| = 2 = \left\lceil \frac{2n}{5} \right\rceil$. Similarly, argument for the cases$n = 10r + 4, 5r, 10r + 7, 10r + 9, 10r + 2;\ r \geq 0$ . On the contrary, if the set of vertices $S = \{v_1, u_2, v_4\}$ is an independent dominating set of $A_5$ & $|S| = 3$ , then it is contradicting the definition of IDN [19]. Consequently, $|S| = 2 = \gamma_i(A_5) = \left\lceil \frac{2n}{5} \right\rceil$. Hence, $\gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for $n \geq 3$.

**Theorem: 3.2** For the antiprism graph $A_n$ the domination number,

$$\gamma(A_n) = \gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil \text{ for } n \geq 3.$$

**Proof:**

From Theorem 3.1, we founded $\gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for $n \geq 3$. According to definition 2.3.1, $S$ is considered an independent dominating set of the antiprism graph$A_n$. This means that, every vertex in $S$ is not adjacent to any other vertex within $S$. Accordingly, $S$ represents the least cardinality dominating set of $A_n$ . Thus, the equivalent sets are the dominating set and the independent dominating set. Therefore, $|S| = \gamma(A_n) = \gamma_i(A_5)$ for $n \geq 3$.

Hence, $\gamma(A_n) = \gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for $n \geq 3$.

**Theorem: 3.3** For the antiprism graph $A_n$ the connected domination number, $\gamma_c(A_n) = (n - 1)$ for $n \geq 3$.

**Proof:**

Let$A_n$ be a 4-regular graph with $n$ faces, consisting of $2n$ vertices and $4n$ edges. It has a girth of 3, indicating that the shortest cycle in the graph has a length of 3. The set $\{u_1, u_2, \ldots, u_n\}$ indicates the vertices on the outer cycle, while $\{v_1, v_2, \ldots, v_n\}$ denotesthe vertices on the inner cycle. According to definition 2.3.2, each vertex within the set $S$ is guaranteed to have atleast one adjacent vertex also belonging to $S$, then $|S|$ is equivalent to the number of vertices in $S$ is equivalent to $n - 1$ for $n \geq 3$. The reasoning for case $n = 3$ is presented, where the set of vertices $S = \{v_1, v_2\}$ is connected dominance number of $A_3$ and $|S| = 2 = n - 1$. Likewise, we can make a similar argument for cases when $n \geq 4$ .

Conversely, if we consider the set of vertices $S = \{v_1, u_2, v_2\}$ as a connected dominating set of $A_3$, where $|S| = 3$, it contradicts the definition of CDN [19]. Consequently, $|S| = 2 = \gamma_c(A_3) = n - 1$. Hence, $\gamma_c(A_n) = (n - 1)$ for $n \geq 3$.

**Theorem: 3.4** For the antiprism graph $A_n(n \geq 3)$, the total domination number,

$$\gamma_t(A_n) = \begin{cases} \left\lceil \frac{4n}{7} \right\rceil & \text{if } n \equiv 0,1,2,3,4,6 \pmod 7 \\ \left\lceil \frac{4n}{7} \right\rceil + 1 & \text{if } n \equiv 5 \pmod 7. \end{cases}$$

**Proof:**

Let $A_n$ is a 4-regular graph with $n$ faces, consisting of $2n$ vertices and $4n$ edges. It has a girth of 3, indicating that the shortest cycle in the graph has a length of 3. The set $\{u_1, u_2, \ldots, u_n\}$ indicates the vertices on the outer cycle, while $\{v_1, v_2, \ldots, v_n\}$ denotes the vertices on the inner cycle. Let $S$ be a total dominating set denoted as $S \subseteq V(A_n)$ for $n \geq 3$, the following two cases are:

**Case (i)** Suppose $n \equiv 0,1,2,3,4,6 \pmod 7$.

According to the clarification of a total dominating set [20], the induced subgraph $< S >$ does not contain any isolated vertices. This means that $S$ contains atleast one neighboring vertex for each vertex. From definition 2.3.3, $S$ contains at least one neighboring vertex for each vertex. Then $|S|$ is equivalent to the number of vertices in $S$ is equivalent to $\left\lceil \frac{4n}{7} \right\rceil$ for $n \geq 7$. The reasoning for case $n = 7$ is presented, where the set of vertices $S = \{v_1, v_2, u_4, u_5\}$ is TDN of $A_7$ and $|S| = 4 = \left\lceil \frac{4n}{7} \right\rceil$ for $n \geq 7$. Similarly argument for the cases $n \equiv 0,1,2,3,4,6 \pmod 7$. On the contrary, if the set of vertices $S = \{v_1, u_1, v_4, u_4, v_7\}$ is a total dominating set of $A_7$ & $|S| = 5$, then it's contradicting the definition of TDN [20]. Consequently, $|S| = 4 = \gamma_t(A_7) = \left\lceil \frac{4n}{7} \right\rceil$ for $n \geq 7$.

Hence, $\gamma_t(A_n) = \left\lceil \frac{4n}{7} \right\rceil$ if $n \equiv 0,1,2,3,4,6 \pmod 7$.

**Case (ii)** Suppose$n \equiv 5 \pmod 7$.

According to the clarification of a total dominating set [20], the induced subgraph $< S >$ does not contain any isolated vertices. This means that $S$ contains atleast one neighbouring vertex for each vertex. From definition 2.3.3, $S$ contains atleast one neighbouring vertex for every vertex. Then $|S|$ is equivalent to the number of vertices in $S$ is equivalent to $\left\lceil \frac{4n}{7} \right\rceil + 1$ for $n \geq 7$. The reasoning for case $n = 12$ is presented, where the set of vertices $S = \{v_1, v_2, u_4, u_5, v_8, v_9, u_{11}, u_{12}\}$ is total domination number of $A_{12}$ and $|S| = 8 = \left\lceil \frac{4n}{7} \right\rceil + 1$ for $n \geq 7$. Similarly argument for the cases $n \equiv 5 \pmod 7$. On the contrary, if the set of vertices $S = \{v_1, v_2, u_3, u_4, v_6, v_7, u_8, u_9, u_{12}\}$ is a total dominating set of $A_{12}$ and $|S| = 9$, then it is contradicting the definition of TDN [20]. Consequently, $|S| = 8 = \gamma_t(A_{12}) = \left\lceil \frac{4n}{7} \right\rceil + 1$ for $n \geq 7$.

Hence, $\gamma_t(A_n) = \left\lceil \frac{4n}{7} \right\rceil + 1$ if $n \equiv 5 \pmod 7$.

**Theorem: 3.5** [19] For the Quartic graph $Q_n(n \geq 10)$, the secure domination number,

$$\gamma_s(G) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + 1 \; if \; n \equiv 0 \; (mod \; 3) \\ \left\lceil \frac{n}{3} \right\rceil \; otherwise \end{cases}$$

**Proof:**

Let G contains set of vertices $\{v_1, v_2, v_3, \ldots \ldots \ldots, v_i, v_{i+1}, v_n\}$. Then all vertices are with degree four. Then $v_1$ vertex is adjacent to $v_2, v_4$ and $v_n, v_{n-1}$. It makes girth four. Therefore any $v_i$ is adjacent to $v_{i+1}, v_{i+3}$ and $v_{i-1}, v_{i-3}$. The proof of the theorem we having the two cases.

**Case (i)** If $n \equiv 0 (mod\ 3)$. In this case, the number of vertices is 6, 9, 12,…,3i, where $i = 1,2, \ldots\ldots, n$. By the construction of quadratic with girth 4 any $v_i$is dominated by $v_{i+1}, v_{i+3,}$ and $v_{i-1}, v_{i-3}$ vertices so at least two dominating vertices need for girth four. But a secure dominating set needs three vertices. The dominating set is at least $\leq 2$ for $n > 9$ and it secures dominating set $S \leq 4$ for $n > 9$ and so on. In this way, we proceed with at least one vertex needed to add for $\left\lfloor \frac{n}{3} \right\rfloor$ vertices in a secure dominating number. Hence $\left\lfloor \frac{n}{3} \right\rfloor + 1 \; if \; n \equiv 0 (mod\ 3)$.

**Case (ii)** Suppose the number of vertices other than $n \equiv 0\ (mod\ 3)$. We used the same method. We need not add one extra vertex if it proves that $\left\lceil \frac{n}{3} \right\rceil$ in all other vertices. It provides the theorem.

**Theorem: 3.6** For the antiprism graph $A_n$ ( $n \geq 3$), the secure domination number, $\gamma_s(A_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if n is odd} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if n is even.} \end{cases}$

**Proof:**

The graph $A_n$ is a 4-regular graph with $n$-faces, consisting of $2n$ vertices and $4n$ edges. It has a girth of 3, indicating that the shortest cycle in the graph has a length of 3. The set $\{u_1, u_2, \ldots, u_n\}$ indicates the vertices on the outer cycle, while $\{v_1, v_2, \ldots, v_n\}$ represents the vertices on the inner cycle. Let $S$ be a secure dominating set $S \subseteq V(A_n)$ for alln $\geq 3$. Then it is discussed in two cases.

**Case (i)** When n is odd.

Let us consider a secure dominating set $S$, where $S$ is a subset of the vertex set $V(A_n)$ for all $n \geq 3$, is defined as follows $S = \{v_i; i = 1, 3, 5, 7, \ldots\ldots, n\}$, and each vertex in S atmost four adjacent vertices in $V - S$. Then $N[S] = N[v_1] \cup N[v_3] \cup N[v_5] \cup \ldots\ldots \cup N[v_n] = V$. Also, $N[v_i] \cap S \leq 2$ for $n \geq 3$. Therefore, $S$ can be regarded as the secure dominating set of $A_n$, and its cardinality, denoted as $|S|$ is equal to $\left\lceil \frac{n}{2} \right\rceil$, where $|S|$ represents the number of vertices in $S$. Thus, $\gamma_s(A_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 3$. Suppose $N[v_i] \cap S = 4$ for any vertex $v_i$ in S, then $|S| = \left\lceil \frac{n}{2} \right\rceil$+1 for $n \geq 3$, it is not satisfied to be definition of SDN [21]. Hence, $\gamma_s(A_n) = \left\lceil \frac{n}{2} \right\rceil$ if n is odd.

**Case (ii)** When n is even.

Let us consider a secure dominating set $S$, where $S$ is a subset of the vertex set $V(A_n)$ for all $n \geq 3$, is defined as follows $S = \{v_i;\ i = 1, 3, 5, 7, \ldots\ldots, n - 1\}$, and every vertex in S at most four adjacent vertices in $V - S$. Then $N[S] =$

$N[v_1] \cup N[v_3] \cup N[v_5] \cup \ldots \ldots \cup N[v_n] = V$. Also, $N[v_i] \cap S \leq 2$ for $n \geq 9$. Therefore, $S$ can be regarded as the secure dominating set of $A_n$, and its cardinality, denoted as $|S|$, is equal to $\left\lfloor \frac{n}{2} \right\rfloor$, where $|S|$ represents the number of vertices in $S$. Thus, $\gamma_s(A_n) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 3$. Suppose $N[v_i] \cap S = 4$ for any vertex $v_i$ in S, then $|S| = \left\lfloor \frac{n}{2} \right\rfloor$+1 for $n \geq 3$, it is not satisfied to be definition of SDN [21]. Hence, $\gamma_s(A_n) = \left\lfloor \frac{n}{2} \right\rfloor$ if n is even.

## 4. DISCUSSION

Within our investigation, we plunge into a range of pivotal inquiries and assumptions surrounding the IDN. Our focus extends to probing the structural characteristics of domination perfect graphs and assessing independent domination outcomes within diverse graph families. Particularly noteworthy is our scrutiny of the maximal ratio between IDN and domination number, an analysis of maximum figures, and the formulation of upper thresholds for connected 4-regular graphs.

Moreover, an intriguing conjecture, as detailed in reference [22], captures our attention. This conjecture postulates $\gamma_i(G) \geq \frac{n}{3}$ remains valid for all connected graphs possessing a measure of at least 4. This proposition presents itself as a compelling subject deserving of thorough investigation.

The visualization depicted in Figure 5 and Figure 6 aptly illustrates the diverse categories of domination parameters. The incorporation of Blue darker solid vertices within these graphical representations serves to indicate any dominating set, offering a visual insight into these intricate concepts.

We noted some significant findings from the graph $Q_n$, as presented below.

(i) When n=7, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have $\gamma(Q_n) = 2 = \gamma_i(Q_n)$=$\gamma_c(GQ_n) = \gamma_t(Q_n)$.

(ii) When n=8, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have $\gamma(Q_n) = 2 = \gamma_c(Q_n)$=$\gamma_t(Q_n)$ but $\gamma_i(Q_n) = 4$.

(iii) When n=9, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have $\gamma(Q_n) = 3 = \gamma_i(Q_n)$=$\gamma_c(Q_n) = \gamma_t(Q_n)$.

(iv) When n=10, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have $\gamma(Q_n) = 2 = \gamma_i(Q_n)$ but $\gamma_c(Q_n) = 4 = \gamma_t(Q_n)$.

(v) When n=11, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have$\gamma$ $(Q_n) = 3 = \gamma_c(Q_n)$=$\gamma_t(Q_n)$ but $\gamma_i(Q_n) = 4$.

(vi) When n=12, the graph $Q_n$ forms a Quartic graph with a girth of 3 and n vertices. In this case, we have $\gamma(Q_n) \leq \gamma_i(Q_n) \leq \gamma_c(Q_n)$=$\gamma_t(Q_n)$.

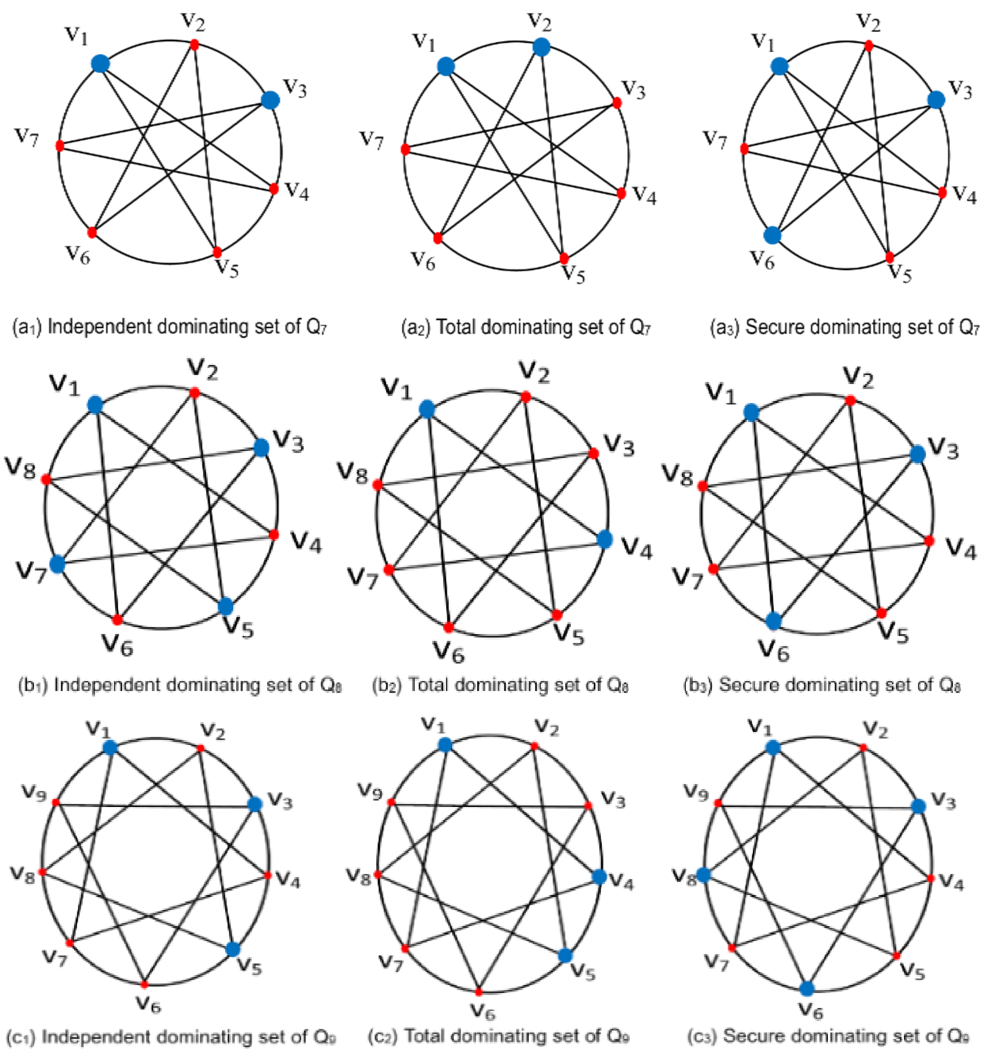
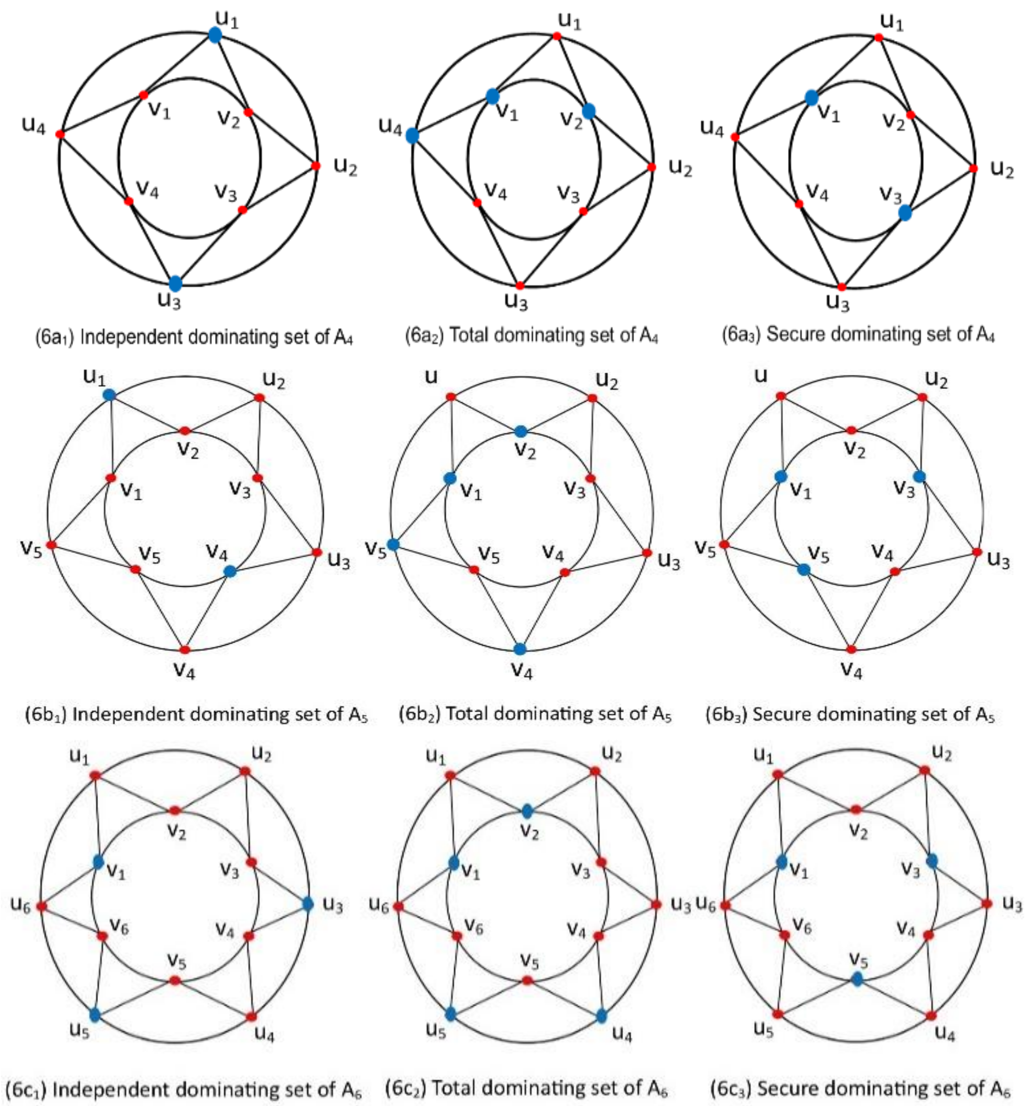



**Figure 5.** Comparing the different types of dominating set of $Q_n$



(6a₁) Independent dominating set of $A_4$ (6a₂) Total dominating set of $A_4$ (6a₃) Secure dominating set of $A_4$

(6b₁) Independent dominating set of $A_5$ (6b₂) Total dominating set of $A_5$ (6b₃) Secure dominating set of $A_5$

(6c₁) Independent dominating set of $A_6$ (6c₂) Total dominating set of $A_6$ (6c₃) Secure dominating set of $A_6$

**Figure 6.** Comparing the different types of dominating set of $A_n$

Novel categories of graphs possessing identical domination and independent domination numbers are presented, accompanied by the precise computation of both their domination and independent domination values [23]. For the antiprism graph $A_n$, then $\gamma_t(A_n) = 2$ for n=3, $\gamma_t(A_n) = 3$ for n=4 and $\gamma_t(A_n) = 4$ for n=5 & 6. Furthermore, when exploring the antiprism graph $A_n$, it can be observed that $\gamma(A_n) = \gamma_i(A_n) = \left\lceil \frac{2n}{5} \right\rceil$ for all $n \geq 3$ and $\gamma_c(A_n) = (n - 1)$ for all $n \geq 3$. Finally, we analysed the antiprism graph $A_n$, then $\gamma_t(A_n) \geq \gamma_s(A_n)$ for all $n \geq 3$. In this paper, we conclude that the inequality is $\gamma_i(A_n) \leq \gamma_t(A_n)$, $\gamma_t(A_n) \leq \gamma_c(A_n)$ and $\gamma_s(A_n) \leq \gamma_t(A_n)$ and also inequalities satisfied the domination chain.

$$\text{(i.e.)}\gamma(A_n) = \gamma_i(A_n) \leq \gamma_s(A_n) \leq \gamma_t(A_n) \geq \gamma_c(A_n)$$

## 5. APPLICATIONS

The concept of the connected domination number holds substantial practical implications in the realm of computer networks, particularly in scenarios where a cohesive cluster of nodes assumes the role of a fundamental communication backbone within the network architecture. This notion gains paramount significance, especially within the domain of mobile networking technology. Furthermore, the notion of domination parameters extends its practical relevance to the domain of coding theory.

The healthcare sector frequently emerges as a prime target for malicious cyber activities, grappling with an alarming frequency of cyberattacks. In this light, the imperative of prioritizing the impregnability of the network infrastructure becomes indisputably clear. This strategic emphasis is indispensable not only to safeguard critical and sensitive information but also to proactively thwart potential threats that could emanate from vulnerabilities in the system [24].

In the intricate landscape of cybersecurity, the strategic integration of a graph-based approach unfolds as an advantageous strategy. Within this approach, diverse graph domination parameters, including but not limited to location-based and secure domination, assume pivotal roles. The insights derived from these parameters contribute substantively to the augmentation of security operations. By harnessing the potential of these parameters, security professionals acquire an enhanced capacity to meticulously scrutinize and bolster their systems against an array of potential threats.

The framework of secure dominating sets can be likened to a network of processors (nodes) orchestrating the secure transmission of sensitive patient data to other designated processors within the system [25]. This construct facilitates seamless remote access for medical practitioners, patients, and their families, all while establishing an impervious bulwark against any unauthorized access attempts orchestrated by malicious hackers. The establishment of secure communication conduits through these dominating sets furnishes a robust platform for the confidential sharing of critical healthcare data among authorized entities [26]. This sophisticated architecture effectively ensures the preservation of data integrity and privacy, maintaining an unwavering stance against potential security breaches.

Central to the attainment of secure communication objectives is the meticulous monitoring of nodes entrenched within the minimum secure dominating set. This targeted concentration on a specific set of nodes conveys the capability to safeguard not only the overall security and coherence of the network but also the sanctity of the information transmitted through it. Through a vigilant oversight of these pivotal nodes, latent vulnerabilities can be promptly identified and efficaciously mitigated, ushering in a state of enhanced security and fortification for the entire system.

When we send a message from one mobile device to another mobile device in an altered signal range, there is a chance that data will be vanished or the message will be sent after an extensive time. These are primarily due to the unstructured or unsystematic manner in which message service systems are located, as well as an unprotected network. The secure domination can be used to overcome these issues. We provide the least or the smallest amount of message centers through secure domination in order to cover and secure the complete block or chain of message centers. To address the aforementioned difficulties, we propose total and secure dominance with a small number of message centers.

## 6. CONCLUSION

This paper contributes to the area of graph protection concept. We have explored multiple variations of vertex domination concepts within the context of Quartic graph and antiprism graph. It is dedicated to the research of the antiprism graph's IDN, CDN, TDN, and SDN. We obtain the bounds for the aforementioned parameters and in particular, for the CDN the bound is $n$-1, for $n \geq 3$. There are several possibilities for future work in secure domination, and SDN of a graph can be determined by adding, deleting or altering edges, as well as deleting vertices and subdivisions.

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