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Investigating Korteweg-de Vries Dynamics via Laplace Transform Methodology

Abdel Baset I. Ahmed[1*], Mohamed A. Labeeb[2]

[1] Engineering Mathematics and Physics Department, Helwan University, Cairo 11511, Egypt
[2] Faculty of Artificial Intelligence, Egyptian Russian University, Cairo 11511, Egypt

Corresponding Author Email: abdel2007@yandex.ru

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ABSTRACT

This study delves into the third-order Korteweg-de Vries equation, a model that encapsulates one-dimensional wave propagation on the water's surface, thereby governing a fully integrable nonlinear system with predetermined initial conditions. The analytical efficacy and efficiency of the Korteweg-de Vries equation have been substantiated. The dynamical behavior of the Korteweg-de Vries equation is examined utilizing the Laplace transform method, which stands out as a direct approach to obtain exact rather than approximate solutions. The existence and uniqueness of precise solutions for the third-order Korteweg-de Vries equation with weak nonlinearity on the semi-axis have been examined. The findings of this work have been extrapolated to another familiar form of the Korteweg-de Vries equation, which describes weakly dispersive waves with minuscule amplitudes in one dimension on the water's surface, and also to the modified strongly dispersive KdV equation.

1. INTRODUCTION

Due to the partial and ordinary differential equations' performance in a wide range of pure and applied mathematics fields, there has been a tremendous amount of research on these topics in the past few years. Differential equations can describe physical models of several occurrences in a variety of domains, and their standards are unbounded. We are well aware that it is extremely difficult to determine the exact solution to such an equation, and that the exact solution's form is frequently too complex to be properly used for numerical calculations. The Laplace transform method, Fourier transform method, and the Green's function method are useful and significant methods to investigate the exact solution of initial value problems for partial and ordinary differential equations [1-4].

The objective of this work was to develop Laplace transform solutions for the Korteweg-de Vries equation (KdV) on the half-line subject to initial conditions. Note that almost all results of this article are new for the KdV equation itself. This equation has numerous applications and is used to represent unlimited wonderment of astrophysical and physical phenomena such as acoustic wave in enharmonic crystals, slightly interacting waves that occurring in shallow water, long internal waves in ocean, ion-acoustic waves that occur in plasma and solitary waves and solitons, that are waves which propagate with the same shape and constant velocity, remaining stable even after mutual collision.

The KdV equation can be expressed in different types, for example:

$$\frac{\partial u}{\partial t}+\alpha\frac{\partial^3 u}{\partial^3 x}+\frac{\partial u^2}{\partial x}=0 \tag{1}$$

introduced by Boussinesq [5] for the first time in 1877, and then rediscovered by Korteweg and De Vries [6] in 1895. The KdV equation was originally developed to explain shallow-water waves, but it has since evolved into an extremely valuable approximation model in nonlinear investigations when weak nonlinearity and weak dispersive equation (its wave solutions spread out in space as they evolve in time) need to be balanced. In which Eq. (2) is now as the fifth-order KdV equation or Kawahara equation.

$$\frac{\partial u}{\partial t}+\beta\frac{\partial^5 u}{\partial^5 x}+\frac{\partial u^2}{\partial x}=0 \tag{2}$$

For the Korteweg-de Vries equation, Bubnov studied the initial-boundary value problem (IBVP) in 1979 [7]. Several IBVPs of the Korteweg-de Vries equation have been the subject of in-depth research since Bubnov's work [8-16].

Early in the new millennium, Bona et al. [8] and Colliander and Kenig [10] respectively established two new, slightly similar techniques to analyse the solvability of the non-homogeneous IBVP of the KdV equation posed on half line. Kenig et al. [17], Kenig et al. [18], and Bourgain [12], investigated the existence and uniqueness solution for Cauchy problem of the KdV equation on the whole line.

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial^3 x}=0,\\ u(x,0)=\emptyset(x),\qquad t,x\in R,\end{gathered} \tag{3}$$

Further, Bona et al. [8, 9] studied the solvability of the KdV Eq. (3) on a finite interval $0<x<1$, $t>0$.

We pay attention to the fact that there are two different strategies for studying IBVPs of dispersive equations that have been established, one by Faminskii (consider a nonlocal mixed problem in a half-strip [11, 19]) and the other by Fokas, Himonas, and Mantzavinos (well-posedness of the KdV equation of IBVPs is established via a fixed point argument in an appropriate solution space [13, 20]).

The paper follows the following sequence. In Section 2, we prove the existence and uniqueness solution of the initial value problem of the nonlinear third-order KdV equation using Laplace transforms and hence through the characteristic equation, we provide the homogeneous solution. Moreover, the particular solution is determined in terms of the Green's function. In Section 3, we provide the most familiar model that controls weakly dispersive waves with tiny amplitudes in one dimension space and classify the solutions of the nonlinear system into homogeneous and particular solution. Then in Section 4, we investigate the Galilean transformed version of the homogeneous nonlinear modified KdV equation. Finally, in conclusion section we list out the obtained solutions as results of using applying Laplace transform method.

Now, we would rather present the nonlinear third-order KdV equation by Laplace transform solutions as a new approach to prove the well-posedness of the initial value problem of the KdV equation.

2. THE REPRESENTATION ON SOLUTIONS OF THE KDV EQUATION BY LAPLACE TRANSFORM

In this paper, through $\mathbb{R}$ we denote the real field. A function $f:(0,\infty)\rightarrow\mathbb{R}$ is said to be of exponential order if there exist constantats $A,B\in\mathbb{R}$ such that $|f(t)|\leq Ae^{Bt}$, for all $t>0$. $f(t)$ has its Laplace transform $\mathcal{L}[f(t)]=F(s)$, where

$$F(s)=\int_0^{\infty} f(t)e^{-st}dt.$$

In which there exists $\delta\in\mathbb{R}$ such that this integral converges if $\Re(s)>\delta$ and diverges if $\Re(s)<\delta$, where $\Re(s)$ is the real part of s. Moreover, $|F(s)|\rightarrow\infty$.

Some properties of Laplace transform can be illustrated as follows

$$\mathcal{L}\left(\frac{df}{dt}\right)=\int_0^{\infty} e^{-st}\frac{df}{dt}dt$$

$$=\left(e^{-st}f(t)\right)\big|_0^{\infty}+\int_0^{\infty} e^{-st}f(t)\,dt.$$

(By integrating by parts)

From convergence conditions of Laplace transform, we assume that

$$\lim_{t\to\infty} e^{-st}f(t)=0.$$

Applying this result, we obtain the following property

$$\mathcal{L}\left(\frac{df}{dt}\right)=s\mathcal{L}(f)-f(0).$$

The derivative property of a time function with respect to a second variable x has the laplace transform:

$$\mathcal{L}\left(\frac{\partial f(t,x)}{\partial x}\right)=\left(\frac{\partial}{\partial x}\right)F(s,x).$$

The integral property of a time function with respect to a second variable x has the laplace transform:

$$\mathcal{L}\left(\int_{x_0}^{x_1} f(t,x)dx\right)=\int_{x_0}^{x_1} F(s,x)dx.$$

Another property of Laplace transform for a function $f(x,t)$ and f_x is a function of x only, then

$$\mathcal{L}(f,f_x)$$
$$=\int_0^{\infty} fe^{-st}.f_x\,dt$$
$$=\left[fe^{-st}\int f_x dt\right]_0^{\infty}-\int_0^{\infty}(f_t-f_x)e^{-st}\left(\int f_x dt\right)dt$$
$$=-f(x,0)\left[\int f_x\,dt\right]_{t=0}-\mathcal{L}\left(f_t.\int f_x dt\right)+\mathcal{L}\left(f_x.\int f_x dt\right)$$

Let's examine how the Laplace transform represents solutions to the nonlinear third-order KdV equation with non-zero initial condition

$$\frac{\partial u}{\partial t}+au\frac{\partial u}{\partial x}+b\frac{\partial^3 u}{\partial^3 x}=f(x,t), \tag{4}$$

$$u(x,0)=\emptyset(x), t\in[0,\infty), x\in R. \tag{5}$$

where, the real-valued function $u(x,t)$ is the average velocity. Waves decay because of the third order term (depressive term), whereas waves steepen because of the nonlinear term.

In the following theorem, we will find the exact solution for the initial-value problem of the nonlinear third-order KdV equation.

Theorem 2.1 The solution $u(x,t)$ of the initial value problem Eqs. (4)-(5) is

$$U(x,s)=A(s)e^{\alpha x}+B(s)e^{\beta x}+C(s)e^{\gamma x}+U_p(x), \tag{6}$$

where, U_p is the particular solution of problem Eqs. (4)-(5) and

$$U_p(x)=\begin{cases}\int_x^{\infty}(G(x,\xi)(f(\xi)d\xi & , \quad \xi<x\\ \int_{-\infty}^{x} G(x,\xi)(f(\xi)d\xi & , \quad \xi>x,\end{cases} \tag{7}$$

$$G(x,\xi)$$
$$=\begin{pmatrix}(\alpha-\gamma)e^{(\alpha+\gamma)\xi+\beta x}\\ +(\beta-\alpha)e^{(\alpha+\beta)\xi+\gamma x}+(\gamma-\beta)e^{(\gamma+\beta)\xi+\alpha x}\end{pmatrix}\Big/ \begin{pmatrix}\alpha(\gamma-\beta)e^{(\beta+\gamma)\zeta+\alpha x}\\ +\beta(\alpha-\gamma)e^{(\alpha+\gamma)\zeta+\beta x}+\gamma(\beta-\alpha)e^{(\alpha+\beta)\zeta+\gamma x}\end{pmatrix}, \tag{8}$$

α,β and γ are roots of the auxiliary equation

$$b\lambda^3+\mathrm{a}u\lambda+\mathrm{s}=0.$$

Proof. Taking Laplace transform with respect to t on both sides of Eq. (4), we have

$$sU(x,s)-u(x,0)+au(x,t)\frac{\partial U(x,s)}{\partial x}+b\frac{\partial^3 U(x,s)}{\partial x^3}=F(x,s).$$

Rearranging the equation, we get

$$b\frac{\partial^3 U(x,s)}{\partial x^3}+au(x,t)\frac{\partial U(x,s)}{\partial x}+sU(x,s)=\emptyset(x)+F(x,s), \quad (9)$$

where, $U(x,s)=\mathcal{L}[u(x,t)]$.

From Eq. (9), yields the characteristic equation of the form

$$b\lambda^3+\mathrm{a}u\lambda+\mathrm{s}=0, \quad (10)$$

which has the roots α, β and γ.

Since the Eq. (9) has a first and third derivatives only with respect to x, then its general solution takes the form

$$U(x,s)=A(s)e^{\alpha x}+B(s)e^{\beta x}+C(s)e^{\gamma x}+U_p(x), \quad (11)$$

where,

$$U_p(x)=\int_x^{\infty}G(x,\xi)f(\xi)d\xi+\int_{-\infty}^{x}G(x,\xi)f(\xi)d\xi$$

is a particular solution of the problem Eqs. (4)-(5) with the Green's function

$$G(x,\xi)=\frac{\Delta}{\Delta'}.$$

where,

$$\begin{aligned}\Delta&=\begin{vmatrix}e^{\alpha\xi} & e^{\beta\xi} & e^{\gamma\xi}\\ \alpha e^{\alpha\xi} & \beta e^{\beta\xi} & \gamma e^{\gamma\xi}\\ e^{\alpha x} & e^{\beta x} & e^{\gamma x}\end{vmatrix}\\
&=e^{\alpha\xi}\left(\beta e^{\beta\xi+\gamma x}-\gamma e^{\gamma\xi+\beta x}\right)-e^{\beta\xi}\left(\alpha e^{\alpha\xi+\gamma x}-\gamma e^{\gamma\xi+\alpha x}\right)\\
&\quad+e^{\gamma\xi}\left(\alpha e^{\alpha\xi+\beta x}-\beta e^{\beta\xi+\alpha x}\right)\\
&=\beta e^{(\alpha+\beta)\xi+\gamma x}-\gamma e^{(\alpha+\gamma)\xi+\beta x}-\alpha e^{(\alpha+\beta)\xi+\gamma x}+\gamma e^{(\beta+\gamma)\xi+\alpha x}\\
&\quad+\alpha e^{(\alpha+\gamma)\xi+\beta x}-\beta e^{(\beta+\gamma)\xi+\alpha x}\\
&=(\alpha-\gamma)e^{(\alpha+\gamma)\xi+\beta x}+(\beta-\alpha)e^{(\alpha+\beta)\xi+\gamma x}\\
&\quad+(\gamma-\beta)e^{(\gamma+\beta)\xi+\alpha x}\end{aligned}$$

and

$$\begin{aligned}\Delta'&=\begin{vmatrix}e^{\alpha\xi} & e^{\beta\xi} & e^{\gamma\xi}\\ \alpha e^{\alpha\xi} & \beta e^{\beta\xi} & \gamma e^{\gamma\xi}\\ \alpha e^{\alpha x} & \beta e^{\beta x} & \gamma e^{\gamma x}\end{vmatrix}\\
&=e^{\alpha\xi}\left(\beta\gamma e^{\beta\xi+\gamma x}-\gamma e^{\gamma\xi+\beta x}\right)-e^{\beta\xi}\left(\alpha\gamma e^{\alpha\xi+\gamma x}-\alpha\gamma e^{\gamma\xi+\alpha x}\right)\\
&\quad+e^{\gamma\xi}\left(\alpha\beta e^{\alpha\xi+\beta x}-\alpha\beta e^{\beta\xi+\alpha x}\right)\\
&=\beta\gamma e^{(\alpha+\beta)\xi+\gamma x}-\beta\gamma e^{(\alpha+\gamma)\xi+\beta x}-\alpha\gamma e^{(\alpha+\beta)\xi+\gamma x}\\
&\quad+\alpha\gamma e^{(\alpha+\beta)\xi+\alpha x}+\alpha\beta e^{(\alpha+\gamma)\xi+\beta x}\\
&\quad-\alpha\beta e^{(\beta+\gamma)\xi+\alpha x}\\
&=(\alpha\gamma-\alpha\beta)e^{(\beta+\gamma)\zeta+\alpha x}+(\beta\alpha-\beta\gamma)e^{(\alpha+\gamma)\zeta+\beta x}\\
&\quad+(\beta\gamma-\alpha\gamma)e^{(\alpha+\beta)\zeta+\gamma x}\\
&=\alpha(\gamma-\beta)e^{(\beta+\gamma)\zeta+\alpha x}+\beta(\alpha-\gamma)e^{(\alpha+\gamma)\zeta+\beta x}\\
&\quad+\gamma(\beta-\alpha)e^{(\alpha+\beta)\zeta+\gamma x}\end{aligned}$$

and α, β and γ are roots of the Eq. (10). Theorem proved.

3. EXAMPLE 1

Since the KdV equation arises in many physical problems, including plasma waves, magnetohydrodynamic waves, and waves with extended wave lengths, it has been taken into consideration.

Consider the Cauchy problem for the important nonlinear KdV equation:

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial^3 x}=0, \quad (12)$$

$$u(x,0)=u_0(x), t\in[0,\infty), x\in R \quad (13)$$

Problem Eqs. (12)-(13) is a model that controls weakly dispersive waves with tiny amplitudes in one dimension and is essential for understanding the soliton concepts. Problem Eqs. (12)-(13) has been investigated by some other methods via the real exponential approach [14], the (G'/G)-expansion method [17], the projective Riccati equation method [18], the tanh-function method [15] and the functional variable method [21].

Now, let us solve the Cauchy problem Eqs. (12)-(13) using Laplace transform method with respect to variable t.

Apply Laplace transform to Eq. (12) taking into account the initial condition Eq. (13), we get

$$\frac{\partial^3 U(x,s)}{\partial x^3}+6u(x,t)\frac{\partial U(x,s)}{\partial x}+sU(x,s)=u_0(x). \quad (14)$$

The characteristic equation of Eq. (14) can be written in the form

$$\lambda^3+6u\lambda+\mathrm{s}=0. \quad (15)$$

Eq. (12) has three roots $\alpha,\beta,$ and γ according to its discriminant

$$32u^3+s^2.$$

(i) If $32u^3+s^2>0$, then the unique real solution of Eq. (15) is obtained from:

$$\alpha=\sqrt[3]{-\frac{s}{2}+\sqrt{\frac{s^2}{4}+8u^3}}+\sqrt[3]{-\frac{s}{2}-\sqrt{\frac{s^2}{4}+8u^3}}.$$

(ii) If $32u^3+s^2<0$, the three real solutions of Eq. (15) are given by:

$$\alpha=2\sqrt{-2u}\cos\left(\frac{Arc\cos\left(\frac{s}{4u}\sqrt{-\frac{1}{2u}}\right)+2\pi}{3}\right)$$

$$\beta=2\sqrt{-2u}\cos\left(\frac{Arc\cos\left(\frac{s}{4u}\sqrt{-\frac{1}{2u}}\right)+4\pi}{3}\right)$$

$$\gamma = 2\sqrt{-2u}\cos\left(\frac{Arc\cos\left(\frac{s}{4u}\sqrt{-\frac{1}{2u}}\right)}{3}\right).$$

(iii) If $32u^3 + s^2 = 0$, then Eq. (15) has three real roots with two the same.

Hence

$$U(x,s) = A(s)e^{\alpha x} + B(s)e^{\beta x} + C(s)e^{\gamma x} + \int_{-\infty}^{+\infty} G(x,\xi)f(\xi)d\xi,$$

$G(x,\xi)$ is calculated in Theorem 2.1.

4. EXAMPLE 2 GALILEAN TRANSFORMED VERSION OF MODIFIED KDV EQUATION (MKDV)

We investigate the Galilean transformed version of the homogeneous nonlinear mKdV equation with initial condition:

$$\frac{\partial u}{\partial t} + 12\epsilon u_0 u\frac{\partial u}{\partial x} + 6\epsilon u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial^3 x} = 0, \epsilon = \pm 1, \quad (16)$$

$$u(x,0) = u_0, \qquad t \in [0,\infty), x \in R \quad (17)$$

where, ϵ indicates whether the equation is focusing or defocusing. This type of equation arises in many physics' problems, such as propagation of ultrashort few-optical cycle solitons in nonlinear media [22, 23] and Alfven waves equation [24], Schottky barrier transmission lines [25], ion acoustic solitons (by adopting an electron equation of state that corresponds to the observed flat-topped electron distribution functions, it is explored how the asymptotic behavior of small ion-acoustic waves depends on the quantity of resonant electrons [26, 27]), thin ocean jets [28], internal waves [29], traffic jam [30], ion acoustic solitons [26], heat pulses in solids [31]. Additionally, both the positive order and the negative order of the mKdV hierarchy illustrate interesting integrable structures [32].

There are numerous standard methods of solution for the mKdV equation's exact solutions for example: communication method [33], Hirota's bilinear method [34], and Wronskian method [35].

It is frequently possible to describe the solutions of a soliton equation with bilinear form using a Wronskian by placing certain restrictions on the entry vector of the Wronskian [36]. Following, we may refer to these conditions as the condition equation set (CES). The coefficient matrix typically plays a key part in the CES of a (1+1)-dimensional soliton problem, and this matrix or any similar shape leads to the same solutions for the related soliton equation. In view of the canonical form of the coefficient matrix, it is thus possible to provide a comprehensive categorization (or structure) for the solutions of the soliton equation [36]. It has been established that, by some limiting process, the solutions derived from a coefficient matrix in Jordan form relate to the solutions derived from a diagonal coefficient matrix [33]. As a result, the previous solutions can be considered limit solutions. According to the IST, N solitons are distinguished from one another by N distinct eigenvalues of the relevant spectral problem, or in other words, N different simple poles $\{k_j\}$ of the transparent coefficient $\frac{1}{a(k)}$. When k_j are multiple-pole solutions, the associated multiple-pole solutions can be derived by restricting the simple-pole solutions using a limiting procedure like $k_2 \to k_1$. The particular solutions in Wronskian form lend themselves more readily to this limiting procedure [33]. A similar process is useful for comprehending the dynamics of limit solutions.

To find the exact solution of problem Eqs. (16)-(17), we use Laplace transforms on Eq. (16), yields

$$sU(x,s) - u_0 + 12\epsilon u_0 u\frac{\partial U(x,s)}{\partial x} + 6\epsilon u^2\frac{\partial U(x,s)}{\partial x} + \frac{\partial^3 U(x,s)}{\partial x^3} = 0,$$

Reorganizing the equation, we get

$$\frac{\partial^3 U(x,s)}{\partial x^3} + 12\epsilon u_0 u\frac{\partial U(x,s)}{\partial x} + 6\epsilon u^2\frac{\partial U(x,s)}{\partial x} + sU(x,s) = u_0 \quad (18)$$

The auxiliary equation of Eq. (18) takes the following form:

$$\lambda^3 + 12\epsilon u_0 u\lambda + 6\epsilon u^2\lambda + s = 0, \quad (19)$$

this equation has three roots three roots $\alpha, \beta,$ and γ.

By virtue of the discriminate

$$32\epsilon^3(2u_0u + u^2)^3 + s^2,$$

there are three cases for the roots

Case i. If $32\epsilon^3(2u_0u + u^2)^3 + s^2 > 0$. Then the unique real solution of Eq. (19) has a unique real solution of the form

$$\alpha = \sqrt[3]{-\frac{s}{2} + \sqrt{\frac{s^2}{4} + 8\epsilon^3(2u_0u + u^2)^3}} + \sqrt[3]{-\frac{s}{2} - \sqrt{\frac{s^2}{4} + 8\epsilon^3(2u_0u + u^2)^3}}.$$

Case ii. If $32\epsilon^3(2u_0u + u^2)^3 + s^2 < 0$. Then Eq. (19) has three real roots as follows:

$$\alpha = 2\sqrt{-2 \in (2u_0u + u^2)}\cos\left(\frac{Arc\cos\left(\frac{s}{4 \in (2u_0u + u^2)}\sqrt{-\frac{1}{2 \in (2u_0u + u^2)}}\right) + 2\pi}{3}\right)$$

$$\beta = 2\sqrt{-2 \in (2u_0u + u^2)}\cos\left(\frac{Arc\cos\left(\frac{s}{4 \in (2u_0u + u^2)}\sqrt{-\frac{1}{2 \in (2u_0u + u^2)}}\right) + 4\pi}{3}\right)$$

$$\gamma = 2\sqrt{-2 \in (2u_0u + u^2)}\cos\left(\frac{Arc\cos\left(\frac{s}{4 \in (2u_0u + u^2)}\sqrt{-\frac{1}{2 \in (2u_0u + u^2)}}\right)}{3}\right)$$

Case iii. If $32u^3 + s^2 = 0$, then Eq. (19) has three real roots with two the same.

Hence, the general solution of problem Eqs. (16)-(17) is

$$U(x,s) = A(s)e^{\alpha x} + B(s)e^{\beta x} + C(s)e^{\gamma x} + \int_{-\infty}^{+\infty} G(x,\xi)f(\xi)d\xi$$

where,

α, β and γ are the roots of the auxillary Eq. (19), $G(x,\xi)$ is also calculated in Theorem 2.1.

5. CONCLUSION

The well-posedness of the initial value problem of the non-linear third-order KdV equations was validated using the Laplace transform method. Through the facilitation of Mathematica, this method can readily be extended to higher-order nonlinear evaluation equations. Evidence suggests that the Laplace transform method serves as a potent and efficient technique for deriving exact solutions for a broad category of problems. An illustrative example was presented to explore the representation of the solution for a familiar class of KdV equations via the Laplace transform. Furthermore, the study encompassed an examination of the Galilean transformed version of the homogeneous nonlinear modified KdV equation.

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