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Examining Captive and Inverse Captive Domination in Selected Graphs and Their Complements

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ABSTRACT

The aim of this paper is to present new properties of captive domination and determine the number of some graphs. The proper subset of the vertices of a graph *G* is a captive dominating set if it is a total dominating set and each vertex in this set dominates at least one vertex which does not belong to the dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set *D* of *G*. If *V-D* contains a dominating set, then this set is called an inverse set of *D* in *G*. The symbol $\gamma^{-1}(G)$ represents the minimum cardinality over all inverse dominating set of *G*. Some graphs which determine the captive domination number such as a ladder graph, corona graph of two paths, lollipop graph, barbell graph, corona graph of a cycle of order n, and null graph of order p and helm graph. For all these graphs and complements the captive domination and inverse captive domination are calculated.

1. INTRODUCTION

Let *G*=(V, E) be a finite simple undirected graph. Let $N(v) = \{u \in V, uv \in E\}$ be the open neighborhood of a vertex *v* and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood set. Also, let $G[D]$ be the subgraph of G induced by vertices in D [1]. A subset $D \subseteq V\ (G)$ is a dominating set of G if $N(v) \cap D \neq \emptyset$; for all $v \in V - D$.

A dominating set *D* is minimal dominating if it does not have a proper subset dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set *D* of *G*. If *V-D* contains a dominating set, then this set is called an inverse set of *D* in *G*. The symbol $\gamma^{-1}(G)$ represents the minimum cardinality over all inverse dominating set of *G* [2]. The concept of domination is used to solve many problems in various fields of mathematics subjects such as topological graphs [2, 3], fuzzy graphs [4, 5], and [6, 7], number theory graph [8], general graphs [9-18], and others. The reader can be found all notions not mentioned [19-22]. The captive domination is initiated by Al-Harere et al. [1], and they get many properties and bounded. the main targeted applications are placing security cameras, or personnel on a site, minimizing the effect of losing a dominating vertex by keeping another vertex with the same capabilities and each dominating vertex is dominating other vertices outside the dominating set-in addition to dominating a neighbouring vertex in the dominating set. In this work, the new properties are discussed, also for some graphs and it is complementing the captive domination and the inverse captive domination are determined for some graphs as a ladder graph, corona graph of two paths, lollipop graph, barbell graph, corona graph of a cycle of order n and null graph of order p, and helm graph.

2. BASIC CONCEPT

Definition 2.1 [1]
Let *G*=(V, E) be a graph without isolated vertices, a subset D ⊂ V(G) is a captive dominating set (CDS) in G if G[D] has no isolated vertex (D is a total dominating set), and each v in set D is adjacent to at least one vertex in *V-D*. The minimum cardinality of a captive dominating set (CDN) of *G* denoted by $\gamma_{ca}(G)$, is called captive domination number.

Observation 2.2
If G is a graph has a CDS, then:
1) Every pendant vertex not belong to each CDS.
2) Every graph has a CDS and pendant vertex has no inverse CDS.

3. CAPTIVE AND INVERSE CAPTIVE DOMINATION IN SELECTED GRAPHS AND THEIR COMPLEMENTS

3.1 Ladder graph

Proposition 3.1.1 If *G* is ladder graph, then:
$\gamma_{ca}(G) = \begin{Bmatrix} 2, \text{if } n = 2 \\ 2\left\lceil \frac{n}{3} \right\rceil, \text{if } n \geq 3 \end{Bmatrix}$ and there is no CDN if *n*=1.

Proof. In the ladder graph, there are two copies of P_n as an induced subgraph. The vertex set of the first path is $\{v_1, v_2, \dots\dots, v_n\}$ and the vertex set of the second path is $\{v_{n+1}, v_{n+2}, \dots\dots, v_{2n}\}$ such that for each vertex in the first path v_i; $1 \leq i \leq n$ there is a corresponding vertex v_{i+n} in the second path and these corresponding vertices are adjacent as shown in the Figure 1.

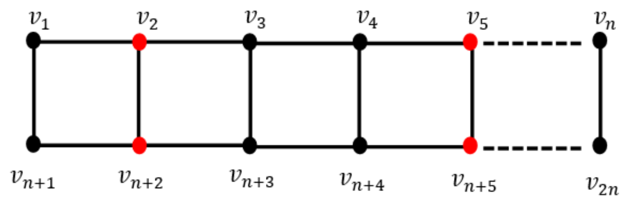

Figure 1. $P_2 \times P_n$

Let $v_1, v_2, \ldots\ldots, v_n, v_{n+1}, v_{n+2}, \ldots\ldots, v_{2n}$ be the vertices of graph G and $D \subset V(G)$ such that:

$$D=\begin{cases} \{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots\ldots, \frac{n}{3} - 1\} \; if \; n \equiv 0 \; (mod\; 3) \\ \{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots\ldots, \left\lceil\frac{n}{3}\right\rceil - 2\} \cup \{v_n, v_{2n}\} \; if \\ \quad n \equiv 1 \; (mod\; 3) \\ \quad n \equiv 2 \; (mod\; 3) \end{cases}$$

For each copy of the path, each vertex can be dominated three vertices as a maximum, so each two corresponding vertices can be *CDS* at most six vertices. Thus, for every consecutive three vertices, the middle vertex can be chosen.

Therefore, depending on *n* there are three classifications as below.

Case 1. If $n = 1$, then the graph $P_2 \times P_1 \equiv P_2$, so one can be concluded that there is no *CDS* in this case.

Case 2. If $n = 2$, then the graph $P_2 \times P_2 \equiv C_4$, and it is obvious that $\gamma_{ca}(P_2 \times P_2) = 2$.

Case 3. If $n \geq 3$, then two subcases that depend on *n* are discussed as the following.

Subcase 1. If $n \equiv 0 \; (mod\; 3)$, then let $D_1 = \left\{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots\ldots, \frac{n}{3} - 1\right\}$, so it is obvious that the set D_1 is *CDS* and it is the minimum, because each vertex dominates the maximum number as possible. Thus, $\gamma_{ca}(G) = \frac{2n}{3}$.

Subcase 2. If $n \not\equiv 0 \; (mod\; 3)$, then two subcases are discussed as the following.

I) If $n \equiv 1 \; (mod\; 3)$, then $n - 1 \equiv 0 \; (mod\; 3)$. In the same manner in Subcaes 1, the set $D_2 = \left\{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots, \left\lceil\frac{n}{3}\right\rceil - 2\right\}$ is minimum *CDS* for all the graph *G* except the two vertices v_n and v_{2n}, so the set $D_3 = D_2 \cup \{v_n, v_{2n}\}$ is the minimum *CDS* and $\gamma_{ca}(G) = 2\left\lceil\frac{n}{3}\right\rceil$.

II) If $n \equiv 2 \; (mod\; 3)$, then $n - 2 \equiv 0 \; (mod\; 3)$.

In the same manner in Subcase 1, the set $D_4 = \left\{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots, \left\lceil\frac{n}{3}\right\rceil - 2\right\}$ is minimum *CDS* for all the graph *G* except the four vertices v_{n-1}, v_n, v_{2n-1}, and v_{2n}, so the set $D_5 = D_4 \cup \{v_n, v_{2n}\}$ is the minimum *CDS* and $\gamma_{ca}(G) = 2\left\lceil\frac{n}{3}\right\rceil$.

From each case above, the required is done.

Proposition 3.1.2 If *G* is ladder graph, then:

$$\gamma^{-1}{}_{ca}(G) = \begin{cases} 2, if \; n = 2 \\ \frac{2n}{3} + 2, if \; n \equiv 0 \; (mod\; 3) \\ 2\left\lceil\frac{n}{3}\right\rceil, if \; n \equiv 1,2 \; (mod\; 3) \end{cases}$$

Proof. Depending on *n* there are three classification as below.

Case 1. If $n = 2$, then the graph $P_2 \times P_2 \equiv C_4$, and it is obvious that $\gamma^{-1}{}_{ca}(P_2 \times P_2) = 2$.

Case 2. If $n \equiv 0 \; (mod\; 3)$, then let $D = \left\{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots\ldots, \frac{n}{3} - 1\right\}$, so it is obvious that the set D is *CDS* and it is the minimum, and let $S = \left\{v_{1+3i}, v_{(n+1)+3i}, i = 0,1, \ldots, \frac{n}{3} - 1\right\} \cup \{v_n, v_{2n}\}$ is inverse *CDS* with respect to D then $\gamma^{-1}{}_{ca}(G) = \frac{2n}{3} + 2$.

Case 3. If $n \equiv 1,2 \; (mod\; 3)$, then let $D_1 = \left\{v_{2+3i}, v_{(n+2)+3i}, i = 0,1, \ldots, \left\lceil\frac{n}{3}\right\rceil - 2\right\} \cup \{v_n, v_{2n}\}$, so it is *CDS* and it is the minimum, and let $S_1 = \left\{v_{1+3i}, v_{(n+1)+3i}, i = 0,1, \ldots, \left\lceil\frac{n}{3}\right\rceil - 2\right\} \cup \{v_{n-1}, v_{2n-1}\}$ is inverse *CDS* with depending on the set D_1 then $\gamma^{-1}{}_{ca}(G) = 2\left\lceil\frac{n}{3}\right\rceil$.

From each case above, the required is done.

Proposition 3.1.3 If G be a graph denoted by $G \equiv \overline{P_2 \times P_n}$ for $n \geq 3$ then $\gamma_{ca}(\overline{P_2 \times P_n})=2$ and there is no captive domination when *n*=1,2.

Proof. Let *G* be a graph of order 2n, depending on *n* there are three classifications as below.

Case 1. If $n = 1$, then the graph $\overline{P_2 \times P_1} \equiv \overline{K_2}$,so one can be concluded that there is no *CDS* in this case.

Case 2. If $n = 2$, then the graph $\overline{P_2 \times P_2} \equiv \overline{C_4}$, so there is no *CDS* in this case.

Case 3. If $n \geq 3$, suppose that the vertices v_1 and $v_{2n} \in P_2 \times P_n$ such that v_1 is adjacent to two vertices v_2, v_{n+1} and the vertex v_{2n} is adjacent to two vertices v_n, v_{2n-1} by proposition 3.1.1 (as shown in the Figure 1) since four vertices in $\overline{P_2 \times P_n}$ has regular degree 2*n*-3 and 2*n*-4 vertices in $P_2 \times P_n$ has regular degree 2*n*-4, (as shown in the Figure 2). therefore, the vertex v_1 is adjacent to all vertices (except the two vertices v_2, v_{n+1}) and the vertex v_{2n} is adjacent to all vertices (except the *two* vertices v_n, v_{2n-1}) in $\overline{P_2 \times P_n}$ the *CDN* is 2.

From each case above, the required is done.

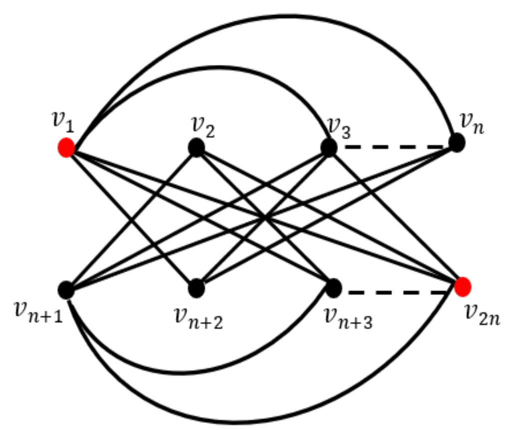

Figure 2. $\overline{P_2 \times P_n}$

Proposition 3.1.4 If G be a graph denoted by $G \equiv \overline{P_2 \times P_n}$ for $n \geq 3$ then $\gamma^{-1}{}_{ca}(\overline{P_2 \times P_n}) = \gamma_{ca}(\overline{P_2 \times P_n}) = 2$ and there is no inverse captive domination when $n = 1,2$.

Proof. Depending on *n* there are three classifications as below.

Case 1. If $n = 1$, then the graph $\overline{P_2 \times P_1} \equiv \overline{K_2}$, so one can be concluded that there is no inverse *CDS* in this case.

Case 2. If $n = 2$, then the graph $\overline{P_2 \times P_1} \equiv \overline{C_4}$, so there is no inverse *CDS* in this case.

Case 3. If $n \geq 3$, by Proposition 2.1 since four vertices in $\overline{P_2 \times P_1}$ has (2*n*-3)-regular degree and 2*n*-4 vertices in $\overline{P_2 \times P_1}$ has (2n-4)-regular degree, then there exist *CDS* in V-D this set is inverse *CDS* respect with *D* then $\gamma^{-1}{}_{ca}(\overline{P_2 \times P_1})$=2.

From each case above, the required is done.

3.2 The corona graph of two paths

Proposition 3.2.1 If *G* be a corona graph denoted by $\mathrm{G} \equiv \mathrm{P_n} \odot \mathrm{P_m}$ then $\gamma_{ca}(\mathrm{G}) = \begin{cases} 2 \text{ if } \mathrm{n} = 1, \mathrm{m} \geq 2 \\ \mathrm{n} \text{ if } \mathrm{n} > 1, \mathrm{m} \geq 1 \end{cases}$, the graph G has no CDS when n=1, m=1.

Proof. Let *G* be a graph of order *n*+*nm* such that P_n be a path of order *n* and P_m be a path of order m then depending on *n* as shown in the Figure 3. There are three classifications as below.

Case 1. If *n=1, m=1* then the graph $P_n \odot P_m \equiv K_2$, so one can be concluded that there is no *CDS* in this case.

Case 2. If $n = 1, m \geq 2$ then the graph $P_n \odot P_m \equiv F_n$, then $\gamma_{ca}(G) = 2$.

Case 3. $n > 1, m \geq 1$, let $v_1, v_2, \dots\dots, v_n$ the vertices of path P_n and let $u_1, u_2, \dots\dots, u_m$ the vertices of path P_m, since every vertex in P_n join with vertices of copy path P_m from u_1 to u_m and $D - v$, $v \in p_n$ not *CDS* then the minimum dominating set D=$\{v_1, v_2, \dots\dots, v_n\}$, and $\gamma_{ca}(G) = n$.

From each case above, the required is done.

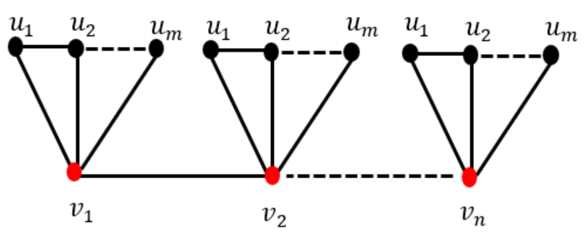

Figure 3. $G \equiv P_n \odot P_m$

Proposition 3.2.2 If G be a corona graph denoted by $\mathrm{G} \equiv \mathrm{P_n} \odot \mathrm{P_m}$ then: $\gamma^{-1}{}_{ca}(\mathrm{G}) = \begin{cases} 2\left\lceil \frac{m}{4} \right\rceil \text{ if } n = 1, m \neq 5{,}9 \\ 2\left\lceil \frac{m}{4} \right\rceil - 1 \text{ if } n = 1, m = 5{,}9 \\ n(2\left\lceil \frac{m}{4} \right\rceil) \text{ if } n \geq 2, m \geq 2 \end{cases}$

and the graph G has no inverse CDS when $n = 1, m = 1{,}2$ and when n=2, m=1.

Proof. Let *G* be a graph of order $n + nm$ such that P_n be a path of order n and P_m be a path of order m then depending on n there are six classifications as below.

Case 1. If *n=1, m=1* then the graph $P_1 \odot P_1 \equiv K_2$, so one can be concluded that there is no inverse *CDS* in this case.

Case 2. If *n=1, m=2* then the graph $P_1 \odot P_2 \equiv K_3$, so one can be concluded that there is no inverse *CDS* in this case.

Case 3. If *n=2, m=1* then the graph $P_2 \odot P_1 \equiv K_3$, so one can be concluded that there is no inverse *CDS* in this case.

Case 4. If $n = 1, m \neq 5{,}9$ and since $P_1 \odot P_m \equiv F_n$, the *CDS D* contains one vertex say v in $P_1$of degree *n-1* such that $D = \{v, u_1\}$, the vertex u_1 be in P_m, so these vertices must be out of inverse *CDS* D^{-1}, then $\gamma_{ca}^{-1}(P_1 \odot P_m) \equiv \gamma_{ca}(P_n)$ by theorem 2.12 [1].

Case 5. If *n*=1, *m*=5,9 then there exist two subcases as follows:

i) If *n=1,m=5* then let $D = \{v_1, u_1\}$, so it is obvious that the set *D* is *CDS* and it is the minimum, so the $D^{-1} = \{u_2 , u_3, u_4\}$ is inverse *CDS* with respect to *D* then $\gamma^{-1}{}_{ca}(G) = 2\left\lceil \frac{m}{4} \right\rceil - 1$.

ii) If *n=1,m=9* then let $D = \{v_1, u_1\}$, so it is obvious that the set *D* is *CDS* and it is the minimum, so the $D^{-1} = \{u_2 , u_3\} \cup \{u_{m-3}, u_{m-2}, u_{m-1}\}$ is inverse *CDS* depending on the set *D* then $\gamma^{-1}{}_{ca}(G) = 2\left\lceil \frac{m}{4} \right\rceil - 1$.

Case 6. If *n*≥2, *m*≥2, now by Proposition 3.1, then the minimum dominating set D=$\{v_1, v_2, \dots\dots, v_n\}$, these vertices must be out of the minimum inverse dominating set D^{-1}, so $\gamma_{ca}^{-1}(P_1 \odot P_m) \equiv n(\gamma_{ca}(P_n))$.

From each case above, the required is done.

Proposition 3.2.3 If G be a graph denoted by $\mathrm{G} \equiv \overline{\mathrm{P_n} \odot \mathrm{P_m}}$ then $\gamma_{ca}(\mathrm{G}) = 2$, is no captive domination when $\mathrm{n} = 1, \mathrm{m} \geq 1$.

Proof. Let *G* be a graph of order *n*+*nm* such that P_n be a path of order *n* and P_m be a path of order *m* then depending on *n* there are two classification as below.

Case 1. If $n = 1, m \geq 1$ then the graph $\overline{P_1 \odot P_m}$ has isolated vertex, so one can be concluded that there is no *CDS* in this case.

Case 2. If $n \geq 2, m \geq 1$ let $v_1, v_2, \dots\dots, v_n$ the vertices of path P_n and let $u_1, u_2, \dots\dots, u_m$ the vertices of path P_m since the vertex u_1 in copy path P_m adjacent to every vertex in P_n and copy path P_m in the graph $\overline{P_1 \odot P_m}$ except two vertices the vertex v_1 and $u_2$whose adjacent with it in $P_n \odot P_m$ and since there exist another vertex in the graph $\overline{P_1 \odot P_m}$ adjacent with u_1 and dominates on these vertices v_1 and u_2 then the *CDN* is 2 (as shown in Figure 4).

Depending to two cases above, the required is done.

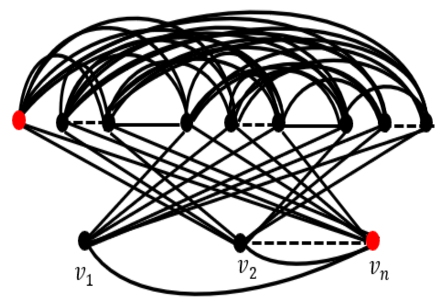

Figure 4. $G \equiv \overline{P_n \odot P_m}$

Proposition 3.2.4 If *G* be a graph denoted by $\mathrm{G} \equiv \overline{\mathrm{p_n} \odot \mathrm{p_m}}$ then $\gamma_{ca}^{-1}(\mathrm{G}) = 2$, the graph *G* has no inverse CDS when $n = 1, m \geq 1$ and *n*=2, *m*=1.

Proof. Depending on *n* there are three classifications as below.

Case 1. If *n*=1, *m*≥1 then the graph $\overline{P_1 \odot P_m}$ has isolate vertex, so one can be concluded that there is no *CDS* in this case and there is no inverse *CDS*.

Case 2. If *n*=2, *m*=1 then the graph $\overline{P_2 \odot P_1} \equiv P_4$, so one can be concluded that there is no inverse *CDS* in this case by Note 3.3 [1].

Case 3. If $n \geq 2$, $m>1$ then by above proposition there exist another dominating set in V-D such that this set is inverse *CDS* and minimum, so $\gamma_{ca}^{-1}(G) = 2$.

From all cases above, the required is done.

3.3 The lollipop graph

Proposition 3.3.1 If G is a lollipop graph denoted by $G \equiv L_{m,n}$ then:

$$\gamma_{ca}(G) = \begin{cases} 2\left\lceil \frac{n}{4} \right\rceil, \text{if } n \equiv 1,2 \pmod 4 \\ 2\left\lceil \frac{n}{4} \right\rceil + 2, \text{if } n \equiv 0,3 \pmod 4 \end{cases}$$

Proof. Let $v_1 = u_0$, then depending on n there are three classifications as below.

Case 1. If $n \equiv 2 \ (mod\ 4)$, then let $D_1 = \left\{u_{4i}, u_{1+4i},\ i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\right\}$. The vertex $u_0 = v_1$, so this vertex dominates all vertices in the induced subgraph which is isomorphic to the complete graph and the vertex u_1 is adjacent to the vertex u_0 and dominates the vertex u_2 as shown in Figure 5. Thus, the two vertices u_0 and u_1 are taken in a *CDS*, after this leave two vertices (u_2 *and* u_3) and take two vertices (u_4 *and* u_5). The vertex u_4 dominates the vertex u_3 and in this way we keep the totality condition of *CDS* and so on. Thus, the set D_1 is a minimum *CDS*.

Case 2. If $n \equiv 1 \ (mod\ 4)$, then depending on n there are two classifications as below.

I) If n=1, then $D = \{v_1, v_2\}$ it is clear that set D is the minimum *CDS*.

II) Let $D_2 = \{v_1, v_2\} \cup \left\{u_{3+4i}, u_{4+4i},\ i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 2\right\}$. two vertices v_1, v_2 so these vertices dominate all vertices in the induced subgraph which is isomorphic to the complete graph and vertex $u_0 = v_1$ is adjacent to the vertex u_1. The two vertices v_1 and v_2 are taken in a *CDS*, after this leave two vertices (u_1 *and* u_2) and take two vertices (u_3 *and* u_4). The vertex u_3 dominates the vertex u_2 and in this way, we keep the totality condition of *CDS* and so on. Thus, the set D_2 is a minimum *CDS*.

Case 3. If $n \equiv 0 \ (mod\ 4)$, then let $D_3 = \{v_1, v_2\} \cup \left\{u_{2+4i}, u_{3+4i},\ i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\right\}$. As the same technique in Case 1, Subcase 2, the set D_3 is a *CDS* of all vertices in the graph.

Case 4. If $n \equiv 3 \ (mod\ 4)$, then let $D_4 = \{v_2, v_3\} \cup \left\{u_{1+4i}, u_{2+4i},\ i = 0, \dots, \left\lceil \frac{n}{4} \right\rceil - 1\right\}$. As the same technique in Case 1, Subcase 2, the set D_4 is a *CDS* of all vertices in the graph.

From all cases above, the required is done.

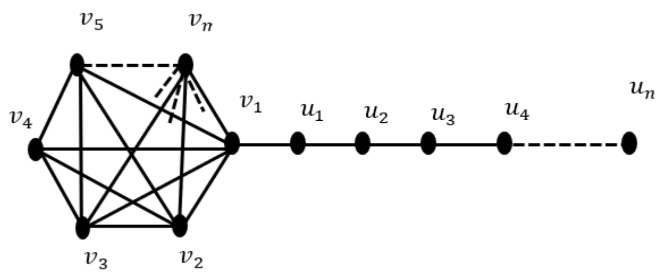

Figure 5. $G \equiv L_{m,n}$

Proposition 3.3.2 If G is a lollipop graph denoted by $G \equiv L_{m,n}$ then G has no *CDN* γ_{ca}^{-1}.

Proof: The graph G has path induced subgraph and according to Observation 2.2, the G has no inverse *CDN*γ_{ca}^{-1}.

Proposition 3.3.3 If $G \equiv \overline{L_{m,n}}$ be a graph then $\gamma_{ca}(G) = 2$, if n>1, and has no *CDS* if n=1.

Proof: Depending on n there are three classifications as below.

I) If n=1, then graph G has an isolated vertex, then there is no *CDS*.

II) Since every vertex in the induced subgraph which is isomorphic to the complete graph dominates on all vertices in the induced subgraph which is isomorphic to the path graph except the vertex v_1 this vertex is adjacent to *n-1* vertices in induced subgraph which is isomorphic to the path graph, then $\gamma_{ca}(G) = 2$.

From all cases above, the required is done.

Proposition 3.3.4 If $G \equiv \overline{L_{m,n}}, n > 1$ be a graph then $\gamma_{ca}^{-1}(G) = 2$.

Proof: By the Proposition above since every vertex in the induced subgraph which is isomorphic to the complete graph dominates on all vertices in induced subgraph which is isomorphic to the path graph except the vertex v_1 this vertex adjacent to n-1 vertices in the induced subgraph which is isomorphic to the path graph then there exist another set in V-D such that this set is minimum and inverse *CDN*.

3.4 The barbell graph

Proposition 3.4.1 If $G \equiv B_{n,n}$ be a barbell graph of order 2n then $\gamma_{ca}(G) = 2$.

Proof: The vertex set of $B_{n,n}$ is $\{v_i : 1 \leq i \leq 2n\}$, the vertex set of the first complete graph K_n is $\{v_1, v_2, \dots \dots, v_n\}$ and the vertex set of the second complete graph K_n is $\{v_{n+1}, v_{n+2}, \dots \dots, v_{2n}\}$ as shown in Figure 6.

From the definition of a barbell graph, there are two copies of a complete graph each of which can dominate by one vertex and to keep the totality of a dominating set, two adjacent vertices are taken. Thus, the set $D = \{v_n, v_{n+1}\}$ is a minimum *CDS*, and $\gamma_{ca}(G) = 2$.

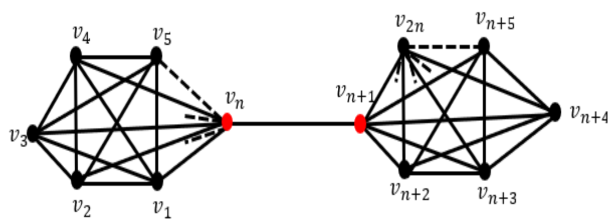

Figure 6. $G \equiv B_{n,n}$

Proposition 3.4.2 If $G \equiv B_{n,n}, n > 2$ be barbell graph of order 2n then $\gamma_{ca}{}^{-1}(G) = 4$.

Proof: By proposition 3.4.1, above let= $\{v_n, v_{n+1}\}$, so it is obvious that the set D is captive dominating and it is the minimum, and since there exists another dominating set in $V - D$ and it is minimum, let $S = \{v_1, v_2\} \cup \{v_{n+2}, v_{n+3}\}$ this set is inverse *CDN* then $\gamma_{ca}{}^{-1}(G) = 4$.

Proposition 3.4.3 If $G \equiv \overline{B_{n,n}}, n > 1$ be a graph of order 2n then $\gamma_{ca}(G) = 2$.

Proof: From the definition of a barbell graph, there are two subgraphs that are isomorphic to a complete graph of order one K_n is $K_1 = \{v_1, v_2, \dots\dots, v_n\}$ and $K_2 = \{v_{n+1}, v_{n+2}, \dots\dots, v_{2n}\}$. In the graph $\overline{B_{n,n}}$ each vertex in the induced subgraph K_1 except the vertex v_n dominates all vertices in the induced subgraph K_2. Also, each vertex in the induced subgraph K_2 except the vertex v_n dominates all vertices in the induced subgraph K_1. Therefore, let $D_1 = \{v_{n-1}, v_{2n-1}\}$ it is obvious that the set D_1 is a captive dominating set and it is minimum. Thus, $\gamma_{ca}(G) = 2$.

Proposition 3.4.4 If $G \equiv \overline{B_{n,n}}, n > 2$ be a graph of order 2n then $\gamma_{ca}{}^{-1}(G) = 2$.

Proof: By Proposition 3.4.3, since the set $S = \{v_1, v_{n+2}\}$ is captive dominating and it is the minimum, then there exists another set in $V - S$ say $W = \{v_2, v_{n+3}\}$, so it is inverse *CDN* and minimum.

3.5 Corona graph of a cycle of order n and null graph of order p

Proposition 3.5.1 If $G \equiv C_n \odot \overline{K_p}$, where C_n be cycle of order n and $\overline{K_p}$ is null graph of order p, then $\gamma_{ca}(G) = n$.

Proof: Each vertex belongs to the induced subgraph which isomorphic to $\overline{K_p}$ not belong to each *CDS* according to Observation 2.2. Moreover, each vertex in the induced subgraph which isomorphic to C_n belongs to each *CDS*. These vertices are totally dominating set and each vertex of them dominates p vertices that represent the set of vertices of a copy of $\overline{K_p}$ as shown in the Figure 7. Thus, $\gamma_{ca}(G) = n$.

Proposition 3.5.2 If $G \equiv C_n \odot \overline{K_p}$ then the graph G has no inverse captive domination number.

Proof: The proof is straightforward according to Observation 1.2.

Proposition 3.5.3 If G be a graph denoted by $G \equiv \overline{K_p} \odot C_n$ then $\gamma_{ca}(G) = 2p$, where p order of $\overline{K_p}$.

Proof: Let $\overline{K_p}$ of order p corona with C_n be a cycle of order n, the graph G denoted by $G \equiv \overline{K_p} \odot C_n$ of order $p(n + 1)$. The vertex set of $\overline{K_p}$ is $\{v_1, v_2, \dots\dots, v_p\}$ and the vertex set of C_n is $\{u_1, u_2, \dots\dots, u_n\}$ as shown in Figure 8.

Graph G consists of p components and each component is isomorphic to a complete graph of order *n+1*, then the *CDS* $D = \{v_p, u_1\}$, and the *CDN* is 2. Thus, $\gamma_{ca}(G) = 2p$.

Proposition 3.5.4 If G be a graph denoted by $G \equiv \overline{K_p} \odot C_n$ then $\gamma_{ca}{}^{-1}(G) = \begin{cases} 2p \; if \; n = 3 \\ 2p \left\lceil \frac{n}{4} \right\rceil \; if \; n > 3 \end{cases}$, where p order of $\overline{K_p}$.

Proof. Depending on the order of the cycle there are two cases as the following.

Case 1. If *n*=3, then each component is isomorphic to a complete graph of order 4, so the order of each of these components is equal to four. Two of these vertices are chosen in the set *D*, so the other two vertices of each component can be chosen to create the other a *CDS* which is disjoint from the set *D*. This set is D^{-1} and it is obvious that is minimum cardinality since $|D^{-1}| = |D| = 2$. Thus, $\gamma_{ca}{}^{-1}(G) = 2p$.

Case 2. *n*>3, then each component is isomorphic to the wheel graph, the center of these components is used in the set D, so we cannot use in another dominating set. Thus, the vertices of the set D^{-1} lie in the induced subgrph isomorphic to the cycle graph. Thus, by using proposition 2.14 [1], $\gamma_{ca}{}^{-1}(G) = 2p \left\lceil \frac{n}{4} \right\rceil$.

From the two cases above, the required is done.

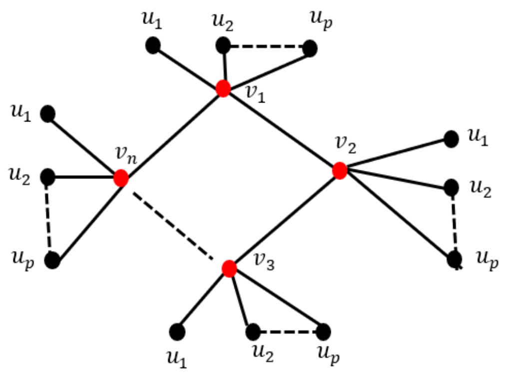

Figure 7. $G \equiv C_n \odot \overline{K_p}$

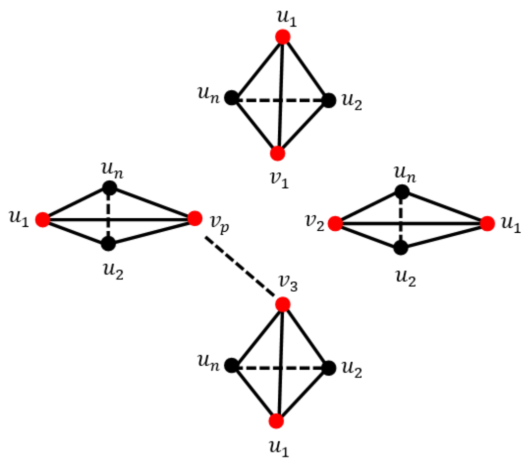

Figure 8. $G \equiv \overline{K_p} \odot C_n$

3.6 The helm graph

Proposition 3.6.1 If $G \equiv H_n$ be helm graph of order $(2n - 1)$ vertices, then $\gamma_{ca}(G) = n - 1$.

Proof. Let G be helm graph of order $(2n - 1)$, then the number of vertices in induced subgraph isomorphic to cycle is $n - 1$ as shown in Figure 9. All these vertices are support vertices by definition must be in *CDS D* then $\gamma_{ca}(G) = n - 1$.

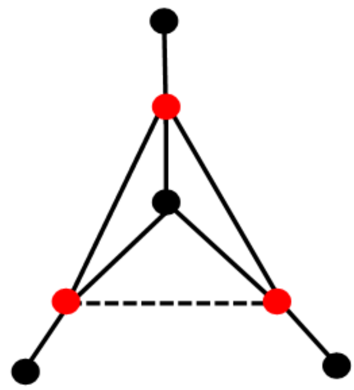

Figure 9. $G \equiv H_n$

Remark 3.6.2 If $G \equiv H_n$ be helm graph of order (2*n*-1) vertices, then G has no inverse *CDS* according to Observation 1.2.

Proposition 3.6.3 If $G \equiv \overline{H_n}$ be a graph of order (2*n*-1) vertices, then $\gamma_{ca}(G) = 2$.

Proof. Each pendent vertex in the helm graph H_n will adjacent to every vertex in $G \equiv \overline{H_n}$ except the support vertex in H_n. Thus, $\gamma_{ca}(G) \geq 2$, now let $D \subseteq \overline{H_n}$ be set contains two pendants' vertices of H_n. It is obvious tht the set D is dominating and the two vertices are adjacent in $\overline{H_n}$, moreover each one of them is adjacent to at least one vertex of the set $V - D$ in $\overline{H_n}$. Therefore, $\gamma_{ca}(G) = 2$.

Proposition 3.6.4 If $G \equiv \overline{H_n}$ be a graph of order $(2n-1)$ vertices, then ${\gamma_{ca}}^{-1}(G) = 2$, if $n \geq 5$ and has no inverse if $n = 4$.

Proof. Depending on n there are two classifications as below.

Case 1. If *n*=4, then by the previous proposition two pendants in H_n make a dominating set in the graph $\overline{H_n}$ (D). In the graph $\overline{H_n}$, the remained pendant vertex in the graph H_n not dominates the support vertex which is adjacent to it in the graph H_n and there is no any vertex in *V-D* dominates this vertex. Thus, the graph $\overline{H_4}$ has no inverse.

Case 2. If *n*>4, then there are at least two pendant vertices not belong to the set *D*. These two pendants' vertices make another dominating set which keep the all conditions of *CDS* and disjoint from the set *D*. Thus, ${\gamma_{ca}}^{-1}(G) = 2$.

From the two cases above, the required is done.

4. CONCLUSIONS

According to the above results, the calculated captive domination of many graphs with it is a compliment, and an inverse of these graphs is a compliment. Most results of these graphs are different. Most results of these graphs are different and we obtained that some graphs have no captive domination number but when we used the operations, we got captive domination like, P_2 have no captive domination number but by using Cartesian product with path P_2 in this case got captive domination equal 2.

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