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Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic Fuzzy Matrices

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ABSTRACT

In this paper, we present the concept of reverse T ordering on Intuitionistic Fuzzy Matrices (IFMs) as an analogue to star ordering on complex matrices. We explore various orderings on IFMs using different generalized inverses, such as g-inverse, group inverse, and Moore-Penrose inverse, and discuss their relationship with reverse T-orderings. By using generalized inverses, we establish some equivalent conditions for each ordering. We demonstrate that the minus ordering constitutes a partial ordering in the set of all regular fuzzy matrices. Moreover, we confirm that these orderings are identical for certain classes of IFMs. Additionally, we elucidate some theorems and examples of reverse T and minus ordering on IFMs using g-inverses.

1. INTRODUCTION

The complexity of problems in Economics, Engineering, Environmental Sciences and Social Sciences which cannot be solved by the well-known methods of classical Mathematics pose a great difficulty in today's practical world. To handle this type of situation Zadeh [1] first introduced the notion of fuzzy set to investigate both theoretical and practical applications of our daily activities. This traditional fuzzy set is sometimes may be very difficult to assign the membership value for fuzzy sets. In the current scenario intuitionistic fuzzy set (IFS) initiated by Atanassov [2] is appropriate for such a situation.

It is well known that generalized inverses exist for a complex matrix. However, this is the failure for fuzzy matrices, that is for $P \in F_{mn}$ under the max-min fuzzy operations the matrix equation PXP=P need not have a solution X. If P has a generalized inverse (g-inverse) then P is said to be regular. The concept of generalized inverse presents a very interesting area of research in matrix theory in the same way a regular matrix as one of which g-inverse exists, lays the foundation for research in fuzzy matrix theory.

Let the IFM A of order m rows and n columns is in the form of $A=[y_{ij},<a_{ija}, a_{ijb}>]$, where a_{ija} and a_{ijb} are called the degree of membership and also the non-membership of y_{ij} in A, it preserving the condition $0\leq a_{ija}+a_{ijb}\leq 1$. In Intuitionistic Fuzzy Matrices, partial ordering is significant. The idea of fuzzy matrix was first presented by Thomosan [3] in 1977 and it has further developments by various researchers. Cen [4] pioneered the partial orderings on fuzzy matrices, which are comparable to the star ordering on complex matrices. After that, a lot of works have been done using this notion. Mitra et al. [5] studied Matrix partial orders. Meenakshi [6] has discussed Fuzzy Matrix: Theory and Applications. Partial ordering is a reflexive, anti-symmetric, transitive crisp binary relation R(X, X). Padder and Murugadas [7] introduced on idempotent Intuitionistic Fuzzy Matrices of T-type. Ben-Israel and Greville [8] Generalized inverses: theory and applications. Kim and Roush [9] have studied Generalized fuzzy matrices. Punithavalli [10] has discussed the partial orderings of m-symmetric fuzzy matrices. Atanassov [11] has studied on the Intuitionistic Fuzzy Implications and Negations. Zheng et al. [12] have discussed Triple I method of approximate reasoning on Atanassov's intuitionistic fuzzy sets. Pradhan and Pal [13] have studied the Generalized Inverse of Atanassov's Intuitionistic Fuzzy Matrices. Padder and Murugadas [14] have characterize Convergence of Power of Controllable Intuitionistic Fuzzy Matrices. Padder and Murugadas [15] have studied Determinant theory for Intuitionistic Fuzzy Matrices. Pradhan and Pal [16] have discussed Some Results on Generalized Inverse of Intuitionistic Fuzzy Matrices. Anandhkumar et al. [17] have studied Pseudo Similarity of Neutrosophic Fuzzy matrices. Anandhkumar et al. [18] have discussed on various Inverse of Neutrosophic Fuzzy Matrices. Anandhkumar et al. [18] have studied Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices. The properties of this class of relations are denoted by the common symbol ≤. Therefore, <x, y> represents <x, y> ∈ R and indicates that x comes before y. The symbol ≥ denotes the reverse partial ordering R^{-1} (X, X). We say that y succeeds x if y≤x implying that <x, y> ∈ R^{-1}. The symbols $\leq^{P}$, $\leq^{Q}$ and $\leq^{R}$ are used to denote the various partial orderings P, Q, and R, respectively. In this Section I, as an analogue to the star ordering on complex matrices, we start with the T inverse or reverse ordering on IFM. We explore different ordering on the IFM using a variety of generalized inverses, including g-inverse, group inverse, and Moore-Penrose inverses, and we analyse how these ordering relate to T ordering. We derive some equivalent conditions for each ordering by using generalized inverses. Additionally, we demonstrate that these orderings are the same for a particular class of IFM. In section II we study the minus ordering for IFM as an analogue of minus ordering for complex matrix studied and as a generalization of T-ordering for IFM introduced. We show that the minus ordering is only a partial ordering in the set of all regular fuzzy matrices. Finally, we characterize the minus ordering on matrix in terms of their generalized inverses.

1.1 Research gaps

As previously introduced, the concept of T and minus ordering on fuzzy matrices was brought forth by Meenakshi, while Cen introduced fuzzy matrix partial orderings and generalized inverses. In this study, we applied the concept of Reverse T and Minus Partial Ordering on Intuitionistic Fuzzy Matrices (IFM). Both these concepts hold significant importance in the context of hybrid fuzzy structures. We delved into the application of Reverse T and Minus Partial Ordering on Intuitionistic Fuzzy Matrices, examining some of the results in depth.

Initially, we offer equivalent characterizations of a Reverse T and Minus Partial Ordering on Intuitionistic Fuzzy Matrices. Subsequently, we derive equivalent conditions for an intuitionistic fuzzy matrix. In addition, using generalized inverses, we further discuss several theorems and provide examples for the reverse T and minus ordering on IFM.

Notations: For IFM of $P \in (IF)_n$,

P^T = Transpose of P,
R(P) = Row space of P,
C(P) = Column space of P,
p^{+}= Moore-Penrose inverse of P,
$(IF)_n$ = Square Intuitionistic Fuzzy Matrices of order n, $(IF)_n^{\#}$ = Intuitionistic Fuzzy group inverse of order n:
$P \leq^T Q$ = T-ordering:
$P \overset{T}{\geq} Q$ = Reverse T-ordering:
$P \overset{\#}{>} Q$ = Reverse Sharp ordering:
$P, Q \in (IF)^{-}_{m\times n}$ = Minus ordering:
$P \leq^{*} Q$ = Partial ordering.

Preliminary and Definitions

Here we recall some preliminary definitions regarding the topic. By a fuzzy matrix, we mean a matrix over a fuzzy algebra. A fuzzy algebra is a mathematical system (F, +,.) with two binary operations addition (+) and multiplication (.) defined on a set F satisfying the following properties:

(P1) Idempotence p+p= p, p. p = p
(P2) Commutativity p+q=q+ p, p.q=q.p
(P3) Associativity p+(q+r)=(p+q)+r, p.(q.r)=(p.q).r
(P4) Absorption (p+p). q=p, p.(p+q)=p
(P5) Distributivity p.(q+r)=(p.q)+(p.q), p+(q.r)=(p+q).(p+r)
(P6) Universal bounds p+0=p, p+1=1; p.0=0, p.1=p. A fuzzy matrix can be interpreted as a binary fuzzy relation, which is defined as follows.

2. REVERSE T-ORDERING ON INTUITIONISTIC FUZZY MATRICES

Definition: 2.1 [4]

For *P, Q* belongs to $(IF)_{m\times n}$ the T-ordering $P \leq^T Q$ is well-defined as $P \leq^T Q \Leftrightarrow P^tP=P^tQ$ and $PP^t=QP^t$.

Definition: 2.2

For *P, Q* belongs to $(IF)_{m\times n}$, the T-Reverse ordering $P \overset{T}{\geq} Q$ is defined as $P \overset{T}{\geq} Q \Leftrightarrow Q^tQ=Q^tP$ and $QQ^t=PQ^t$.

Example: 2.1 Let us consider,

$$P = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <1,0> \end{bmatrix}, Q = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}.$$

Theorem 2.1

Let $P, Q \in (IF)_{m\times n}$ and Q^+ exists. Then the given conditions are equivalent.

(i) $P \overset{T}{\geq} Q$
(ii) $Q^+Q=Q^+P$ and $QQ^+=PQ^+$
(iii) $QQ^+P=Q=PQ^+Q$

Proof: (i) ⇒ (ii), By (i) we have $P \overset{T}{\geq} Q \Leftrightarrow Q^t Q=Q^tP$ and $QQ^t=PQ^t$

Then $Q^+Q=Q^+QQ^+Q=Q^+(Q^+)^t Q^t Q=Q^+(Q^+)^t Q^t P=Q^+Q Q^+P=Q^+P$

Similarly, $QQ^+=PQ^+$

(ii) ⇒ (iii) $Q^+Q=Q^+P$ implies $Q=QQ^+Q=QQ^+P$ and $QQ^+=PQ^+$

implies $Q=QQ^+Q=PQ^+Q$

(iii) ⇒ (i) By $Q=QQ^+P$, $(QQ^+)^t Q=(QQ^+)^t P$

Then, $Q^t (Q^t)^t Q^t Q=Q^t (Q^+)^t Q^t P$. Hence $Q^t Q=Q^t P$

Similarly, $QQ^t = PQ^t$ by $Q=PQ^+Q$.

Theorem 2.2

Let $P, Q \in (IF)_{m\times n}$ if P^+ and Q^+ both exists. Then the given conditions are equivalent.

(i) $P \overset{T}{\geq} Q$
(ii) $Q^+Q=P^+Q$ and $QQ^+=QP^+$
(iii) $P^+QQ^+=Q^+=Q^+QP^+$
(iv) $Q^t Q P^+=Q^t=P^+ Q Q^t$

Proof: (i) ⇒ (iv) $Q^t Q=Q^t P$ implies $Q^t Q=Q^t PP^+P$.

Then $Q^tQ=(Q^tQ)^t=(P^+P)^t(Q^tP)^t=P^+PQ^tQ$.

Hence, $Q^tQQ^+=P^+PQ^tQQ^+$ and $Q^t (QQ^+)^t=P^+PQ^t (QQ^+)^t$.

Therefore, $Q^t=P^+PQ^t=P^+QQ^t$.

Similarly, $Q^t=Q^tQP^+$ by $QQ^t=PQ^t$.

(iv) ⇒ (ii) By $Q^t=P^+QQ^t$, $Q^t (Q^+)^t=P^+QQ^t(Q^+)^t$.

Then, $Q^+Q=P^+QQ^+Q=P^+Q$.

Similarly, $QQ^+=QP^+$ and $Q^t=Q^tQP^+$.

(ii) ⇒ (i) $Q^+Q=(Q^+Q)^t=(P^+Q)^t=(P^+PP^+Q)^t=(P^+Q)^t(P^+P)^t =(Q^+Q)^tP^+P=Q^+QP^+P=Q^+QQ^+P=Q^+P$.

Similarly, we have $QQ^+=PQ^+$. Thus (i) holds by Theorem 2.1 (ii).

(ii) ⇒ (iii) By $Q^+Q=P^+Q$, $Q^+=Q^+QQ^+=P^+QQ^+$.

Similarly, $QQ^+=QP^+$ implies $Q^+=Q^+QP^+$.

(iii) ⇒ (ii) $P^+QQ^+=Q^+=Q^+QP^+$ implies $Q^+Q=P^+QQ^+Q$ $=P^+Q$ and $QQ^+=QQ^+QP^+=QP^+$.

Theorem 2.3

In $(IF)^+_{m\times n}$, the set of all IFM $P \in (IF)_{m\times n}$ for which P+exists $\overset{T}{\geq}$ is a partial ordering.

Proof: $R \overset{T}{\geq} R$ obvious. If $R \overset{T}{\geq} Q$, $Q \overset{T}{\geq} R$ then $R=QR^+R$, $P=PP^+R$ by theorem 2.1 (iii). Thus, by Theorem 2.2 (ii) $P=PP^+Q=PQ^+Q=Q$.

If $R \overset{T}{\geq} Q$,$Q \overset{T}{\geq} P$ then $R=QR^+R$ and $Q=PQ^+Q$ by Theorem 2.1 (iii) and Theorem 2.2 (ii), we have $R=QR^+R=PQ^+QR^+R=PQ^+R=PR^+R$.

Similarly, we have $R=RR^+P$. Thus, $R \overset{T}{\geq} P$ by Theorem 2.1 (iii).

Example: 2.2

Let $Q = \begin{bmatrix} <1,0> & <0,1> \\ <1,0> & <1,0> \end{bmatrix}, P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$.

For Q, $QQ^t \neq Q$. Therefore Q^+ does not exists. Here $Q \overset{T}{\geq} P$ and $P \overset{T}{\geq} Q$ but, $Q\neq P$. Thus $\overset{T}{\geq}$ is not a partial ordering in $(IF)_{m\times n}$.

Theorem 2.4

If $P \overset{T}{\geq} Q$ then we have:

(i) $P^+Q=Q^+P$ and $QP^+=PQ^+$.

(ii) $P^tQ=Q^tP$ and $PQ^t=QP^+$ (i.e) P^tQ and PQ^t are symmetric,

(iii) $QP^+Q=Q=PP^+Q=PQP^+=P^+QP$, $PQ^+Q=Q=QQ^+P=QPQ^+=Q^+PQ$.

(iv) $PQ^tQ=QQ^tP=Q^tPQ=QPQ^t$, $QP^tP=PP^tQ=P^tQP=PQQ^t$.

Theorem 2.5 If $P \overset{T}{\geq} Q$, then we have:

(i) $P^t \overset{T}{\geq} Q^t$.

(ii) $P^+ \overset{T}{\geq} Q^+$.

(iii) $P^tQ \overset{T}{\geq} P^tP, QP^t \overset{T}{\geq} QQ^t$.

(iv) $P^+Q \overset{T}{\geq} P^+P, QP^+ \overset{T}{\geq} PP^+$.

(v) $P^tP \overset{T}{\geq} Q^tQ, PP^t \overset{T}{\geq} QQ^t$.

(vi) $P^+P \overset{T}{\geq} Q^+Q, PP^+ \overset{T}{\geq} QQ^+$.

(vii) If $P^tP^+=P^+P^t$ then $Q^tQ^+=Q^+Q^t$.

(viii) if $P^+=P^t$ then $Q^+=Q^t$.

(ix) if $P^2=0$ then $Q^2=0$.

(x) if $P=P^2$ then $Q=Q^2$.

(xi) if $P=PP^T$ then $Q=QQ^t$.

(xii) if $P=P^T=P^3$ and $Q=Q^t$ then $Q=Q^3$.

Proof: (i) and (ii) hold clearly.

(iii) $(P^tQ)^tP^tQ=Q^tPP^tQ=Q^tQQ^tQ=Q^tQQ^tP=Q^tQP^tP=Q^tPP^tP$.

Similarly, $P^tQ(Q^tQ)^t=P^tP(P^tQ)^t$, thus $P^tQ\overset{T}{\geq}P^tP$.

Similarly, we have, $QP^t\overset{T}{\geq}PP^T$.

(iv) $(P^+Q)^tp^+Q=(Q^+Q)^tQ^+Q=Q^t(Q^+)^tQ^+Q=Q^t(Q^+)^tQ^+P$ $=Q^t(Q^+)^tP^+P=(Q^+Q)^tP^+P=(P^+Q)^tP^+P$ and $P^+Q(P^+Q)^T=P^+P(P^+Q)^T$. Thus $P^+Q\overset{T}{\geq}P^+P$.

Similarly, we have $QP^+\overset{T}{\geq}PP^+$.

(v) $P^tP\overset{T}{\geq}Q^tQ$,

$P^tP \overset{T}{\geq} Q^tQ, PP^t \overset{T}{\geq} QQ^t$

$(Q^tQ)^tQ^tQ = Q^tQQ^tP = Q^tQP^tP = (Q^tQ)^tP^tP$ and $Q^tQ(Q^tQ)^t = P^tP(Q^tQ)^t$

$P^tP \overset{T}{\geq} Q^tQ$.

Similarly, $PP^t \overset{T}{\geq} QQ^t$.

(vi) $P^+P \overset{T}{\geq} Q^+Q, PP^+ \overset{T}{\geq} QQ^+$

$(Q^+Q)^tP^+P = Q^t(Q^+)^tP^+P = Q^t(Q^+)^tQ^+P$ $= Q^t(Q^+)^tQ^+Q = (Q^+Q)^tQ^+Q$ and $P^+P(Q^+Q)^t = Q^+Q(Q^+Q)^t$

($Q^tQ=Q^tP$ and $QQ^t=PQ^t$)

(vii) If $P^tP^+=P^+P^t$ then $Q^tQ^+=Q^+Q^t$

$Q^tQ^+=Q^+QP^tP^+QQ^+=Q^+QP^+P^tQQ^+=Q^+Q^t$

(viii) if $P^+=P^t$ then $Q^+=Q^t$

$Q^+=Q^+\ Q\ Q^+=P^+\ QQ^+=P^+\ QP^+=P^+QP^t=P^+QQ^t=Q^t$

(ix) if $P^2=0$ then $Q^2=0$

$Q^2=QP^+PPP^+Q=QP^+P^2\ P^+Q=0$

(x) if $P=P^2$ then $Q=Q^2$

$Q^2=QP^+PPP^+Q=QP^+PP^+Q=QP^+Q=QQ^+Q=Q$

(xi) if $P=PP^t$ then $Q=QQ^t$

By $P=PP^t$, we have $P^t=P$ and $P^+=P$

Then $QQ^t=PQ^t=PP^tQ=PPQ^t=PQQ^t=P^+QQ^t=Q^t=Q$

(xii) $Q^3=QQ^tQ=PQ^tQQ^+P=PP^tQQ^+P=PP^tPQ^+P=PQ^+P=PP^+Q=Q$.

3. REVERSE MINUS ORDERING ON IFM

Definition: 3.1

For $P \in (IF)^-_{m,n}$ and $Q \in (IF)_{m\times n}$the inverse or Reverse minus ordering as $\geq$is defined as $P \geq Q \Leftrightarrow Q^-Q = Q^-P$ and $QQ^- = PQ^-$ for some $Q^- \in Q\{1\}$.

To specify the minus ordering with respect to particular g-inverse of P, let us write $P \geq Q$ with respect to $X \Leftrightarrow XQ = XP$ and $QX = PX$ for $X \in Q\{1\}$.

Remark: 3.1

For $Q \in (IF)^-_{m,n}$ and $P \in (IF)_{m\times n}$ if Q^+ exists, then Q^+ is unique and $Q^+=Q^T$ we have, $P \overset{T}{\geq} Q \Leftrightarrow P \geq Q$ with respect to $Q^+ \Leftrightarrow Q^tQ = Q^tP$ and $QQ^t = PQ^t$, which is precisely Definition 2.1 of T-ordering. Thus T-ordering is a special case of minus ordering. However, the converse $P \geq Q \Rightarrow P \overset{T}{\geq} Q$ need not be true.

Example: 3.1

Let us consider, $P = \begin{bmatrix} <1,0> & <1,0> \\ <0,0> & <0,0> \end{bmatrix}, Q = \begin{bmatrix} <1,0> & <1,0> \\ <0,0> & <0,0> \end{bmatrix}$. Since Q^t is a g-inverse of Q, Q^+ exist and $Q^+=Q^t$ also Q is idempotent, Q itself is a g-inverse of Q, $Q=QP=PQ$ implies $P \geq Q$ with respect to Q. $Q^tQ \neq Q^tP$ and $QQ^t \neq PQ^t$. Hence, $P \geq Q$ not implies $P \overset{T}{\geq} Q$.

Theorem: 3.1

For $Q \in (IF)^{-}{}_{m,n}$ and $P \in (IF)_{m\times n}$ the given conditions are equivalent

(i) $P \geq Q$

(ii) $Q = QQ^{-}P = PQ^{-}Q = PQ^{-}P$

Proof: (i) ⇒ (ii)

$P \geq Q \Leftrightarrow Q^{-}Q = Q^{-}P$ and $QQ^{-} = PQ^{-}$ for some $Q^{-} \in Q\{1\}$

Now, $Q = Q(Q^{-}Q) = QQ^{-}P$

$Q = (QQ^{-})Q = PQ^{-}Q$

$Q = P(Q^{-}Q) = PQ^{-}P$

(ii) ⇒ (i)

Let $X = Q^{-}QQ^{-}$

$QXQ = Q(Q^{-}QQ^{-})Q = (QQ^{-}Q)Q^{-}Q = Q \Rightarrow X \in Q\{1\}$

Now, $XQ = (Q^{-}QQ^{-})QQ^{-}P = Q^{-}(QQ^{-}Q)Q^{-}P = (Q^{-}QQ^{-})P = XP$

Similarly, *QX=PX*

Hence $P \geq Q$ with respect to $X \in Q\{1\}$.

Remark: 3.2

In general, in the definition of minus ordering $P \geq Q$, P need not be regular. This is illustrated in the following example.

Example: 3.2

Let us consider,

$$P = \begin{bmatrix} <1,0> & <1,0> & <0,0> \\ <0,0> & <1,0> & <1,0> \\ <0,0> & <0,0> & <1,0> \end{bmatrix}, \quad Q = \begin{bmatrix} <1,0> & <1,0> & <1,0> \\ <1,0> & <1,0> & <1,0> \\ <1,0> & <1,0> & <1,0> \end{bmatrix}$$

Since Q is idempotent, Q is regular and Q itself is a g-inverse of Q. Here *Q=QP=PQ*. Hence $P \geq Q$ which implies $Q = Q^{-} \in Q\{1\}$. If P is not regular, since there is no $X \in F_3$ such that *PXP=P*.

Theorem: 3.2

Let $P, Q \in (IF)^{-}{}_{m,n}$. If $P \geq Q$, then $P\{1\} \subseteq Q\{1\}$.

Proof: $P \geq Q \Rightarrow Q = QQ^{-}P = PQ^{-}Q$

For, $P^{-} \in P\{1\}$

$QP^{-}Q = (QQ^{-}P)P^{-}(PQ^{-}Q)$

$QP^{-}Q = QQ^{-}(PP^{-}P)Q^{-}Q = (QQ^{-}P)Q^{-}Q = QQ^{-}Q = Q$

Hence, $QP^{-}Q = Q$ for each $P^{-} \in P\{1\}$

Therefore, $P\{1\} \subseteq Q\{1\}$.

Theorem: 3.3

If $P \geq Q$ and Q is idempotent then Q is a g-inverse of P.

Proof. Let P itself is a *g*-inverse of P then P is regular, P is idempotent. Here $P \in P\ \{1\}$. Then by above property $P\{1\} \subseteq Q\{1\}$. Hence P is a g-inverse of Q.

Example: 3.3

Let us consider,

$$P = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,1> \end{bmatrix}, Q = \begin{bmatrix} <1,0> & <1,0> \\ <0.5,0.5> & <0,1> \end{bmatrix}$$

P is not idempotent.

$Q\{1\} = \left\{X: X = \begin{bmatrix} <1,0> & <\beta,0> \\ <1,0> & <\alpha,0> \end{bmatrix}, 0.5 \leq \beta \leq 1, \text{and } 0 \leq \alpha \leq 1\right\}$

Here, $P \geq Q$ for $Q^{-} = \begin{bmatrix} <0,1> & <0.5,0> \\ <1,0> & <1,0> \end{bmatrix}$ but $P \notin Q\{1\}$.

Theorem 3.3

For $P, Q \in (IF)^{-}{}_{m,n}$ then the given conditions are equivalent.

(i) $P \geq Q$

(ii) $Q = QP^{-}P = PP^{-}Q = QP^{-}Q \; for\ all\ P^{-} \in P\{1\}$

(iii) $R(Q) \subseteq R(P), C(Q) \subseteq C(P)$ and $QP^{-}Q = Q$.

Proof: (i) ⇒ (ii): $Q = PQ^{-}P$ (By Theorem 3.1) $= PQ^{-}(PP^{-}P) = (PQ^{-}P)P^{-}P = QP^{-}P$ (By Theorem 3.1)

Therefore, $Q = QP^{-}P$for each $P^{-} \in P\{1\}$

Similarly, we have $Q = PP^{-}Q$ for each $P^{-} \in P\{1\}$ (By Theorem 3.1)

(ii) ⇒ (iii): $Q = QP^{-}P = PP^{-}Q = QP^{-}Q \; for\ all\ P^{-} \in P\{1\}$

$Q = QP^{-}P$ for all $P^{-} \in P\{1\}$

$Q = XPP^{-}P$, *Q=XP*

$Q = XP \Leftrightarrow R(Q) \subseteq R(P)$

$Q = PP^{-}Q$ for all $P^{-} \in P\{1\}$

$Q = PP^{-}PY$

$Q = PY \Leftrightarrow C(Q) \subseteq C(P)$,

(iii) ⇒ (i): Let $X = P^{-}QP^{-}$

$QXQ = Q(P^{-}QP^{-})Q$

$QXQ = (QP^{-}Q)P^{-}Q = QP^{-}Q = Q \Rightarrow X \in Q\{1\}$

Now, $QX = Q(P^{-}QP^{-}) = PP^{-}Q(P^{-}QP^{-}) = PP^{-}(QP^{-}Q)P^{-} = PP^{-}QP^{-} = PX$.

Similarly, *XQ=XP* and $QP^{-}Q = Q$

Hence $P \geq Q$ with respect to $X \in Q\{1\}$.

Theorem 3.4

For $(IF)^{-}{}_{m,n}$ the minus ordering ≥ is a partial ordering.

Proof: (i) $R \geq R$ is obvious. Hence ≥ is reflexive.

(ii) $R \geq Q \Rightarrow R = QR^{-}Q$

$Q \geq R \Rightarrow Q = QQ^{-}R = RQ^{-}Q$

$R = QR^{-}Q = (QQ^{-}R)R^{-}(RQ^{-}Q) = QQ^{-}(RR^{-}R)Q^{-}Q = QQ^{-}(RQ^{-}Q) = QQ^{-}Q = Q$.

Thus, $R \geq Q$ and $Q \geq R \Rightarrow R = Q$. Hence ≥ is antisymmetric.

(iii) $R \geq Q \Rightarrow R = RQ^{-}A$ and $R = RQ^{-}Q = QQ^{-}R$

$Q \geq P \Rightarrow Q = QQ^{-}P = PQ^{-}Q$

Let $X = Q^{-}RQ^{-}$. Then $RXR = R(Q^{-}RQ^{-})R = (RQ^{-}R)Q^{-}R = RQ^{-}R = R$.

Since, $R \geq Q$ and $Q \geq P$ applying Theorem 3.2 repeatedly, we have $RX = R(Q^{-}RQ^{-}) = QQ^{-}R(Q^{-}RQ^{-}) = QQ^{-}(RQ^{-}R)Q^{-} = QQ^{-}RQ^{-} = (PQ^{-}Q)Q^{-}RQ^{-} = PQ^{-}(QQ^{-}R)Q^{-} = P(Q^{-}RQ^{-}) = PX$.

Similarly, *XR=XP*. Since $X \in R\{1\}$ with *RX=PX* and *XR=XP* it follows that, $R \geq P$.

Theorem 3.5

For $P \in (IF)^{-}{}_{m,n}$ and $Q \in (IF)_{m\times n}$ the given conditions are equivalent:

(i) $P \geq Q \Leftrightarrow P^t \geq Q^t$.

(ii) $P \geq Q \Leftrightarrow RPS \geq RQS$ for some invertible matrices *R* and *S*.

Proof: $P \geq Q \Leftrightarrow QQ^- = PQ^- and Q^-Q = Q^-P \Leftrightarrow (QQ^-)^t = (PQ^-)^t \Leftrightarrow (Q^-)^tQ^t = (Q^-)^tP^t \Leftrightarrow (Q^t)^-Q^t = (Q^t)^-P^t$.

$QQ^- = PQ^- \Leftrightarrow (Q^t)^-Q^t = (Q^t)^-P^t$.

Similarly, $Q^-Q = Q^-P \Leftrightarrow Q^t(Q^t)^- = P^t(Q^t)^-$.

Hence, $P \geq Q \Leftrightarrow P^t \geq Q^t$.

(ii) $P \geq Q \Leftrightarrow RPS \geq RQS$ for some invertible matrices *R* and *S* $P \geq Q \Leftrightarrow QQ^- = PQ^- and\ Q^-Q = Q^-P$.

Since *P* is regular which implies *RPS* is also regular and $S^TP^-R^T$ a g-inverse of *RPS*.

$(RQS)^-(RQS) = (S^tQ^{-1}R^t)SQR$

$(RQS)^-(RQS) = S^tQ^{-1}(R^tR)QS$

$(RQS)^-(RQ) = S^t(Q^{-1}Q)S$

$(PDQ)^-(PDQ) = S^t(Q^{-1}P)S$

$(PDQ)^-(PDQ) = (S^tQ^{-1}R^t)(RPS)$

$(PDQ)^-(PDQ) = (RQS)^-(RPS)$

Similarly, $(RQS)(RQS)^- = (RPS)(RQS)^-$.

Hence, $P \geq Q \Rightarrow (RPS) \geq (RQS)$.

Conversely, $(RPS) \geq (RQS) \Rightarrow R^t(RPS)S^t \geq R^t(RQS)S^t \Rightarrow P \geq Q$.

Corollary: 3.1

For $P, Q \in (IF)^{+}{}_{m,n}$, $P \geq Q$ with respect to $P^+ \Leftrightarrow P^+ \geq Q^+$ with respect to C.

Theorem 3.6

For $P \in (IF)^{-}{}_{m,n}$ and $Q \in (IF)_{m\times n}$ with $P \geq Q$.

(i) If $P = P^2$, then $Q = Q^2$.

(ii) If $P^2 = 0$, then $Q^2 = 0$.

Proof: $Q^2 = QQ = (QQ^-P)(PQ^-Q) = QQ^-P^2Q^-Q = (QQ^-P)Q^-Q = QQ^-Q = Q$

$Q^2 = QQ = (QQ^-P)(PQ^-Q) = QQ^-P^2Q^-Q = 0$.

Remark: 3.3

In the above Theorem 3.1, if $P \geq Q$ with *Q* idempotent then *P* need not be idempotent. Consider $P = \begin{bmatrix} <0,0> & <1,0> \\ <1,0> & <0,0> \end{bmatrix}, Q = \begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <1,0> \end{bmatrix}$. Here $P \geq Q$ with respect to $Q^- = Q$. But *P* is not idempotent.

Theorem 3.7

For $P, Q \in (IF)^{+}{}_{m,n}$, $P \overset{T}{\geq} Q \Leftrightarrow P \geq Q$ and $PQ^+P = Q$.

Proof: $P \overset{T}{\geq} Q$ and by remark (3.1) it follows that $P \geq Q$ and $QQ^+B = Q \Rightarrow Q = PQ^+P$.

Conversely: if $P \geq Q$ by Theorem 3.3 $Q = QP^-Q$ for all $P^- \in P\{1\}$. Since $P \in (IF)^{+}{}_{m,n}$, P^+ exist and $P^t=P^+$ is a g-inverse of *P*, hence $Q=QP^+Q=QP^tQ$.

Now, $QQ^t = Q(PQ^+P)^t$ $QQ^t = QP^tQP^t = (QP^tQ)P^tQQ^t = QP^t$.

Thus, $QQ^t = QP^t \Rightarrow QQ^t = PQ^t$ (Taking transpose on both sides).

Similarly, we have $Q^tQ = Q^tP$.

Hence, $P \overset{T}{\geq} Q$.

Theorem 3.8

For $P, Q \in (IF)^{+}{}_{m,n}$, the following conditions are equivalent.

i. $P \geq Q$ with respect to $Q^+ \left(P \overset{T}{\geq} Q\right)$.

ii. $P \in Q^+\{1,3,4\}$.

iii. $P^+ \in Q\{1,3,4\}$.

Proof: (i) $\Rightarrow$ (ii) $P \geq Q$ with respect to $Q^+ \Rightarrow Q^+Q = Q^+P$ and $QQ^+ = PQ^+$.

Now, $Q^+ = Q^+QQ^+ = Q^+PQ^+ \Rightarrow P \in Q^+\{1\}$ $(Q^+P)^t = (Q^+Q)^t = Q^+Q = Q^+P \Rightarrow P \in Q^+\{3\}$ $(PQ^+)^t = (QQ^+)^t = QQ^+ = PQ^+ \Rightarrow P \in Q^+\{4\}$.

(ii) $\Rightarrow$ (iii) Since $Q^+ = Q^t$ and $P^+ = P^t$, we have , $P \in Q^+\{1,3,4\} \Rightarrow P^+ \in Q\{1,3,4\}$.

(iii)$\Rightarrow$ (i) $P^+ \in Q\{1,3,4\} \Rightarrow QP^+Q = Q, (QP^+)^t = QP^+ = PQ^+$ and $(P^+Q)^t = P^+Q = Q^+P$ $Q^+Q = Q^+Q(P^+Q) = (Q^+QQ^+)P = Q^+P$ $QQ^+ = (QP^+Q)Q^+ = (QP^+)QQ^+ = PQ^+QQ^+ = PQ^+$.

Hence $P \geq Q$ with respect to Q^+.

Theorem 3.9

For $P, Q, R \in (IF)^{+}{}_{m,n}, R \in P\{2\}$ and $Q \geq R$ then $Q \in P\{2\}$.

Proof: $Q \geq R \Rightarrow RR^-Q = QR^-R = QR^-Q = Q$ $QPQ = (QR^-R)P(RR^-Q) = QR^-(RPR)R^-Q = QR^-(RR^-Q) = QR^-Q = Q$.

Hence, $Q \in P\{2\}$.

4. CONCLUSION

In this study, we derive equivalent conditions for each ordering by leveraging generalized inverses. We also demonstrate that these orderings coincide for a specific class of Intuitionistic Fuzzy Matrices (IFM). Our exploration of the minus ordering for IFM serves as an analogue to the minus ordering for complex matrices studied previously and as an extension of the T-ordering for IFM introduced earlier.

The distinction between this research and prior work lies in the characterization of the Reverse Tilde and Minus Partial Ordering on Intuitionistic Fuzzy Matrices, as contrasted with the previous study's focus on Sharp and Left-T and Right-T partial ordering. We established that for commuting pairs of matrices, the Reverse Tilde and minus ordering are equivalent. We further demonstrate that under certain conditions, the Reverse Tilde simplifies to the T-ordering on IFM.

We present a set of necessary conditions for IFMs with specified row and column spaces to fall under sharp ordering. We introduce the concept of Reverse Tilde orderings for IFM as an analogue of complex matrices, showing that these orderings preserve the Moore-Penrose inverse property. By employing various generalized inverses, we discuss a new type of minus ordering. Finally, we demonstrate that these orderings are identical for a certain class of IFM.

In future research, we plan to prove some related properties of the generalized inverse of Reverse Tilde and Minus Partial Ordering on Intuitionistic Fuzzy Matrices.

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