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A Variational Formulation for Coupled Single-Walled Carbon Nanotubes Undergoing Vibrations in the Presence of an Axial Magnetic Field

Sarp Adali

Discipline of Mechanical Engineering, University of KwaZulu-Natal, Durban 4041, South Africa

Corresponding Author Email: adali@ukzn.ac.za

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ABSTRACT

A variational formulation is presented for a system consisting of two coupled single-walled carbon nanotubes subjected to forced vibrations, longitudinal magnetic field, and axial compression, based on the nonlocal elasticity theory. The variational principle for the double nanotube system is derived, followed by the application of Hamilton's principle to express kinetic and potential energies. Subsequently, the time-independent scenario is investigated, and governing equations for the freely vibrating system are provided. The variational formulation for this case is established, and expressions for Rayleigh quotients concerning the vibration frequency and buckling load are derived. The Rayleigh quotient for the frequency demonstrates that the magnetic field increases the vibration frequency of the coupled nanotube system. Nonlocal effects appear in both the numerator and the denominator of the Rayleigh quotient, influencing the frequency increase or decrease depending on the relative values of various problem parameters. In contrast, the magnetic field reduces the buckling load, as evidenced by its negative contribution to the numerator of the Rayleigh quotient for buckling. The effect of the nonlocal parameter on buckling, however, cannot be inferred directly from the Rayleigh quotient. In this study, involving a system of two coupled partial differential equations, it is crucial to derive variationally consistent boundary conditions. Utilizing the formulated variational principle, variationally consistent natural boundary conditions are established in terms of moment and shear force expressions. It is revealed that the Pasternak interlayer between the nanotubes results in coupled boundary conditions when a shear force and/or a moment is specified at the boundaries of the nanotube system.

1. INTRODUCTION

Magnetic fields play a significant role in various nanotechnology applications, particularly in nano and micro electromechanical systems and nanosensors. Carbon nanotubes (CNTs) exposed to magnetic fields exhibit unique features, which can be harnessed for numerous nanotechnology applications as reported by Kibalchenko et al. [1]. The influence of magnetic fields on the vibration frequencies of CNTs was discussed by Zhang et al. [2], suggesting that magnetic fields can be utilized to control the dynamic characteristics of CNTs. Moreover, magnetic fields affect wave propagation in nanotubes with implications for fluid conveyance in CNTs, as noted by Arani et al. [3]. Alazwari et al. [4] observed that exposure to a magnetic field induces a hardening effect, increasing system stiffness and resulting in higher frequencies. This effect can be employed for resonance control in nanosensors, as reported by Kiani [5].

In the realm of biomedicine, magnetic fields offer advantages such as the development of non-invasive and harmless medical instruments like MRIs, as mentioned by Samadishadlou et al. [6]. Another medical application involves the use of CNTs with ferromagnetic nanoparticles for hyperthermia studies when exposed to a magnetic field, as studied by Raniszewski et al. [7]. Several studies by Bellucci et al. [8], Klinovaja et al. [9], Kibalchenko et al. [10], Kiani [11], and Fedorov et al. [12] have investigated the effect of magnetic fields on CNTs and CNT devices, providing valuable insights for the development of various nanodevices. For instance, Popov et al. [13] devised a magnetic field sensor, while Mandal et al. [14] and Pal et al. [15] explored magnetic nanomotors.

Recent investigations have delved into the magnetic properties of nanotubes and the behavior of nanotubes in a magnetic field. Bellucci et al. [16] examined the impact of magnetic fields on the transport properties of nanotubes, while Kibalchenko et al. [17] investigated their electronic properties. Furthermore, Li et al. [18] and Wang et al. [19] studied the influence of magnetic fields on the dynamics of multiwalled nanotubes. The mechanics of CNTs in a magnetic field have been extensively researched, with studies by Wang et al. [20, 21], Kiani [22], and Narendar et al. [23] focusing on the effect of a longitudinal magnetic field on wave propagation in single-walled carbon nanotubes (SWCTs).

More recent studies on wave propagation in SWCTs exposed to magnetic fields include works by Arani et al. [24, 25] involving fluid-conveying single and double-walled nanotubes, and Li et al. [26], which considered wave propagation in viscoelastic single-walled nanotubes while incorporating the surface effect using strain gradient theory. Zhen and Zhou [27] investigated wave propagation in fluid-conveying nanotubes under the influence of magnetic fields, considering thermal and surface effects. Vibrations of carbon nanotubes exposed to magnetic fields were studied for double-walled nanotubes by Kiani [28, 29] and Murmu et al. [30], and for double single-walled carbon nanotube systems by Murmu et al. [31] and Arani et al. [32]. Alazwari et al. [4] examined vibrations of functionally graded nanobeams in a magnetic field, while Kiani [33] investigated vibration and buckling of CNTs in a three-dimensional magnetic field.

Jena et al. [34] and Esen et al. [35] studied the effect of magnetic fields on vibrations and buckling of nanobeams. Nonlinear vibrations of nanobeams and nanotubes in a magnetic field were investigated by Chang [36], Ebrahimi and Hosseini [37], and Anh and Hieu [38]. Jalaei et al. [39] examined the dynamic stability of a Timoshenko nanobeam in a magnetic field, while Bahaadini et al. [40] focused on viscoelastic nanotubes conveying magnetic nanoflow in a magnetic field. These studies primarily explored wave propagation and linear/nonlinear vibrations of single-walled nanotubes and nanobeams under specific boundary conditions.

Complex mechanical systems formed by multiple carbon nanotubes, particularly those involving double single-walled nanotubes (DSWNTs), have garnered significant attention in recent years due to their distinct mechanical properties from double-walled nanotubes. A plethora of research has focused on studying the vibrational behavior of DSWNTs and double nanobeams connected by an elastic layer in the presence of a magnetic field, as evidenced by the works of Kiani [28, 29], Murmu et al. [30, 31], Arani et al. [32], Nasirshoaibi et al. [41], and Stamenkovic et al. [42]. Ebrahimi and Dabbagh [43] extended these findings by investigating the effect of magnetic fields on the propagation of acoustical waves in rotary double-nanobeam systems.

The current study aims to develop a variational formulation for a double single-walled carbon nanotube (SWCNT) system subjected to forced vibrations, an axial magnetic field, and axial compression, based on nonlocal elasticity theory. To date, a rigorous derivation of variational principles and associated boundary conditions for such double CNT systems exposed to magnetic fields remains unavailable. By providing a variational framework, the present study paves the way for future investigations into the vibration and buckling behavior of CNTs in magnetic fields.

Variational formulations have been previously explored in numerous cases involving carbon nanotubes and nanobeams, including those by Yang and Lim [44], Baretta et al. [45], Barretta et al. [46], and Chalamel [47]. Ike [48] presented a formulation based on the system's energy for flexural torsional buckling of open cross-section columns, and proposed a variational formulation for a Timoshenko beam undergoing bending in a subsequent study [49].

Deriving variationally consistent boundary conditions is crucial for complex mechanical systems. Several studies have focused on this topic, such as those by Shi and Voyiadjis [50] for shear deformable beams, Robinson and Adali [51] for carbon nanotubes subjected to uniformly and triangularly distributed axial loads, and Barretta et al. [52] for Timoshenko nanobeams based on nonlocal strain gradient theory. Further examples include research by Yu et al. [53] on size-dependent beams, Xu and Zheng [54] on Timoshenko nanobeams, and Pinnola et al. [55] on nonlocal gradient elastic beams. Barretta et al. [56] obtained variationally consistent boundary conditions for nonlocal strain gradient Timoshenko nanobeams.

Adali [57, 58] derived variational principles and variationally consistent boundary conditions for multi-walled carbon nanotubes experiencing buckling and vibrations, respectively. Xu and Deng [59] extended these findings to natural and geometric boundary conditions for multi-walled nanotubes subject to buckling and vibrations based on strain gradient theory. Additional research on this topic includes work by Kucuk et al. [60] on vibrating multi-walled nanotubes modeled as Timoshenko beams and Adali [61] on buckling. Adali [62] also formulated variational principles and variationally consistent boundary conditions for a double Rayleigh beam system based on local elasticity theory.

In this study, we present a variational principle for a double nanotube system subjected to a magnetic field, based on nonlocal elastic theory. By identifying the kinetic and potential energies in the variational expression, we derive Hamilton's principle. We then focus on the freely vibrating system, formulating the corresponding variational principle and deriving Rayleigh quotients for frequency and buckling load. The presence of a magnetic field is shown to increase the frequency, while decreasing the buckling load. Finally, we derive natural and geometric boundary conditions and provide expressions for shear force and moment.

The remainder of this paper is structured as follows: In Section 2, we develop the variational principle for a double SWCNT system connected by a Winkler-Pasternak interlayer, subjected to forced vibrations, axial magnetic field, and compressive force. We then present the variational formulation of the problem, derive Hamilton's principle based on the variational formulation, and obtain expressions for Rayleigh quotients for vibration frequency and buckling load. In Section 3, we derive variationally consistent boundary conditions in terms of moment and shear force expressions, which are essential for variational and approximate solution methods, particularly those involving the Rayleigh-Ritz method. Examples of such methods applied to Kirchhoff plates can be found in Ike [63] and Nwoji et al. [64], while applications to nanotubes and nanobeams are presented in studies by Ansari et al. [65], Chakraverty and Behera [66], Behera and Chakraverty [67], and Fakher and Hosseini-Hashemi [68]. In Section 4, we provide a summary and conclude the paper.

2. GOVERNING EQUATIONS

Double single-walled nanotube system under consideration is subject to a longitudinal magnetic field H_x acting in the axial direction and this results in the Lorentz force $\eta A_i H_x^2$ acting in the transverse direction as noted in Stamenkovic et al. [42]. In the expression $\eta A_i H_x^2$, η is the magnetic field permeability and A_i is the cross sectional area of the i^{th} nanotube. Nanotubes 1 and 2 are subject to axial compressive forces P_1 and P_2 and continuous transverse forces $f_i(x, t)$ (i=1, 2), respectively, with the external forces acting on the beams in the time domain $t_1 \leq t \leq t_2$.

The nanotubes are connected by an elastic layer between them which is modelled as a combination of a Winkler layer with an elastic modulus of k_0 and a Pasternak (shear) layer with an elastic modulus of G_0 as shown in Figure 1.

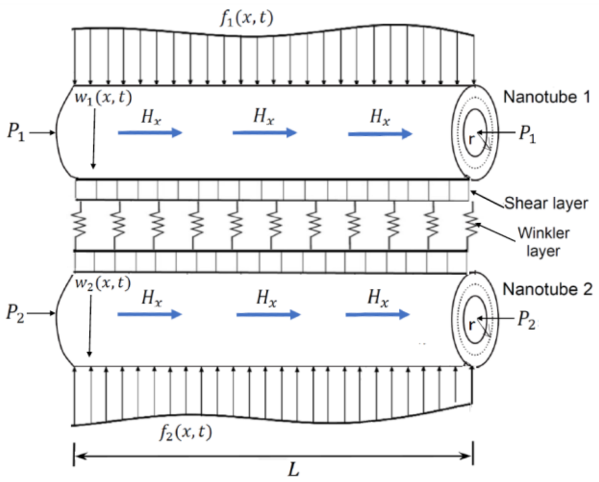

Figure 1. Double single-walled carbon nanotubes with Winkler and shear layer subject to axial magnetic field

The deflections of the nanotubes are given by $w_1(x, t)$ and $w_2(x, t)$ with the elastic stiffnesses denoted by EI_i, cross-sectional areas by A_i and the densities by ρ_i. The equations governing the forced vibrations of the double nanotubes subject to a magnetic field are given in Stamenkovic et al. [42] based on the Euler–Bernoulli beam and nonlocal elasticity theories detailed in Eringen [69, 70]. These equations can be expressed as follows:

$$L_1(w_1) - \mu N_1(w_1) + K(w_1,w_2) = f_1(x,t) - \mu \frac{\partial^2 f_1(x,t)}{\partial x^2} \tag{1}$$

$$L_2(w_2) - \mu N_2(w_2) - K(w_1,w_2) = f_2(x,t) - \mu \frac{\partial^2 f_2(x,t)}{\partial x^2} \tag{2}$$

where, the differential operators $L_i(w_i)$ and the coupling operators $K_i(w_1, w_2)$ are given by:

$$L_i(w_i) = EI_i \frac{\partial^4 w_i}{\partial x^4} + P_i \frac{\partial^2 w_i}{\partial x^2} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} - \eta A_i H_x^2 \frac{\partial^2 w_i}{\partial x^2} \tag{3}$$

$$N_i(w_i) = P_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^4 w_i}{\partial x^2 \partial t^2} - \eta A_i H_x^2 \frac{\partial^4 w_i}{\partial x^4} \tag{4}$$

$$K(w_1,w_2) = k_0(w_1 - w_2) - G_0\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2}\right) - \mu k_0\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2}\right) + \mu G_0\left(\frac{\partial^4 w_1}{\partial x^4} - \frac{\partial^2 w_2}{\partial x^4}\right) \tag{5}$$

It is noted that in Eqs. (1) and (2), $\mu = (e_0 a)^2$ is the nonlocal parameter as defined in Eringen [69, 70]. The derivations of the Eqs. (1)-(5) are given in Stamenkovic et al. [42] with the formulations of the magnetic field acting on the nanotubes based on the works of Murmu et al. [30, 31].

3. VARIATIONAL FORMULATION

In the present section, the variational formulation of the problem is developed. In order to formulate the variational principle applicable to the system of Eqs. (1)-(2), we first introduce the variational functionals $V_1(w_1, w_2)$ and $V_2(w_1, w_2)$ as follows

$$V[w_1,w_2] = V_1[w_1,w_2] + V_2[w_1,w_2] \tag{6}$$

where, $V(w_1, w_2)$ is the variational functional to be determined. The variational functionals $V_1(w_1, w_2)$ and $V_2(w_1, w_2)$ are expressed as follows:

$$V_i[w_1,w_2] = \Phi_i[w_i] - \mu\Psi_i[w_i] + \Lambda_W[w_1,w_2] + \Lambda_P[w_1,w_2] - w_i f_i + \mu w_i \frac{\partial^2 f_i}{\partial x^2} \tag{7}$$

where the functionals $\Phi_i[w_i]$, $\Psi_i[w_i]$, $\Lambda_W[w_1,w_2]$ and $\Lambda_P[w_1,w_2]$ are defined as follows:

$$\Phi_i[w_i] = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \begin{bmatrix} E_i I_i \left(\frac{\partial^2 w_i}{\partial x^2}\right)^2 - P_i\left(\frac{\partial w_i}{\partial x}\right)^2 \\ -\rho_i A_i \left(\frac{\partial w}{\partial t}\right)^2 + \eta A_i H_x^2 \left(\frac{\partial w}{\partial x}\right)^2 \end{bmatrix} dx\, dt \tag{8}$$

$$\Psi_i[w_i] = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \begin{bmatrix} P_i\left(\frac{\partial^2 w_i}{\partial x^2}\right)^2 + \rho_i A_i \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial t^2} \\ -\eta A_i H_x^2 \left(\frac{\partial^2 w_i}{\partial x^2}\right)^2 \end{bmatrix} dx\, dt \tag{9}$$

$$\Lambda_W[w_1,w_2] = \frac{k_0}{2}\int_{t_1}^{t_2}\int_0^L \begin{bmatrix} (w_1 - w_2)^2 \\ +\mu\left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x}\right)^2 \end{bmatrix} dx\, dt \tag{10}$$

$$\Lambda_P[w_1,w_2] = \frac{G_0}{2}\int_{t_1}^{t_2}\int_0^L \begin{bmatrix} \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x}\right)^2 \\ +\mu\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2}\right)^2 \end{bmatrix} dx\, dt \tag{11}$$

Thus we have:

$$V_1(w_1,w_2) = \Phi_1[w_1] - \mu\Psi_1[w_1] + \Lambda[w_1,w_2] - w_1\left(f_1 - \mu\frac{\partial^2 f_1}{\partial x^2}\right) \tag{12}$$

$$V_2(w_1,w_2) = \Phi_2[w_2] - \mu\Psi_2[w_2] + \Lambda[w_1,w_2] - w_2\left(f_2 - \mu\frac{\partial^2 f_2}{\partial x^2}\right) \tag{13}$$

where, $\Lambda[w_1, w_2]$ is defined as follows:

$$\Lambda[w_1,w_2] = \Lambda_W[w_1,w_2] + \Lambda_P[w_1,w_2] \tag{14}$$

It is observed that the Euler-Lagrange equations of the variational functional $V(w_1, w_2)$ given by Eq. (6) corresponds to the governing equations of the double nanotube system given by Eqs. (1)-(2). This can be verified by taking the first variation of the functional $\delta V[w_1, w_2]$ with respect to w_1 and w_2.

4. HAMILTON'S PRINCIPLE

The Hamilton's principle can be expressed as:

$$\int_{t_1}^{t_2}\left[\delta KE(t) - \left(\delta W_E(t) + \delta PE_1(t) + \delta PE_2(t)\right)\right]dt = 0 \tag{15}$$

In the present problem, the functionals $KE(t)$, $W_E(t)$, $PE_1(t)$ and $PE_2(t)$ are given by:

$$KE(t) = \frac{1}{2}\sum_{i=1}^{2}\int_0^L \left[\rho_i A_i \left(\frac{\partial w_i}{\partial t}\right)^2 + \mu\rho_i A_i \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial t^2}\right] dx \tag{16}$$

$$W_E(t) = -\sum_{i=1}^{2}\int_0^L \left(f_i - \mu\frac{\partial^2 f_i}{\partial x^2}\right) w_i(x,t)\, dx \tag{17}$$

$$PE_1(t) = \frac{1}{2}\sum_{i=1}^{2}\int_0^L \begin{bmatrix} E_i I_i \left(\frac{\partial^2 w_i}{\partial x^2}\right)^2 - P_i\left(\frac{\partial w_i}{\partial x}\right)^2 \\ +\eta A_i H_x^2 \left(\frac{\partial w_i}{\partial x}\right)^2 \end{bmatrix} dx \tag{18}$$

$$PE_2(t) = \frac{1}{2}\sum_{i=1}^{2}\int_0^L \begin{bmatrix} k_0(w_1 - w_2)^2 + \mu k_0\left(\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial x}\right)^2 \\ +G_0\left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x}\right)^2 \\ +\mu G_0\left(\frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_2}{\partial x^2}\right)^2 \end{bmatrix} dx \tag{19}$$

In Eqs. (16)-(19), $KE(t)$ is the kinetic energy of the double nanotube system, $W_E(t)$ is the work done by external forces and $PE_1(t)$ is the potential energy of deformation which includes the contributions by the compressive forces P_i and the magnetic field $\eta A_i H_x^2$. $PE_2(t)$ is the potential energy due to the Winkler and Pasternak layers between the nanotubes.

5. FREE VIBRATIONS

Variational principle for a freely vibrating double beam system is formulated next. In this case the external forces $f_i(x,t)=0$ with i=1, 2. For freely vibrating beams, the deflection can be expressed as:

$$w_i(x,t) = W_i(x)\, e^{i\omega t} \tag{20}$$

where, ω is the vibration frequency. The differential equations for the freely vibrating double nanotube system can be obtained from the system of Eqs. (1)-(5) and can be expressed as:

$$L_{FV1}(W_1) - \mu N_{FV1}(W_1) + K(W_1, W_2) = 0 \tag{21}$$

$$L_{FV2}(W_2) - \mu N_{FV2}(W_2) - K(W_1, W_2) = 0 \tag{22}$$

In Eqs. (21)-(22), the differential operators $L_{FVi}(W_i)$ and $N_{FVi}(W_i)$ are given by:

$$L_{FVi}(W_i) = E_i I_i \frac{d^4 W_i}{dx^4} + P_i\frac{d^2 W_i}{dx^2} - \omega^2\rho_i A_i W_i - \eta A_i H_x^2 \frac{d^2 W_i}{dx^2} \tag{23}$$

$$N_{FVi}(W_i) = P_i\frac{d^4 W_i}{dx^4} - \omega^2\rho_i A_i\frac{d^2 W_i}{dx^2} - \eta A_i H_x^2\frac{d^4 W_i}{dx^4} \tag{24}$$

The coupling operator $K(W_1, W_2)$ is defined as:

$$K(W_1, W_2) = k_0(W_1 - W_2) - G_0\left(\frac{d^2 W_1}{dx^2} - \frac{dW_2}{dx^2}\right) - \mu k_0\left(\frac{d^2 W_1}{dx^2} - \frac{d^2 W_2}{dx^2}\right) + \mu G_0\left(\frac{d^4 W_1}{dx^4} - \frac{d^2 W_2}{dx^4}\right) \tag{25}$$

For the freely vibrating case, the variational functional $V_{FV}[w_1, w_2]$ is of the form:

$$V_{FV}[W_1, W_2] = V_{FV1}[W_1, W_2] + V_{FV2}[W_1, W_2] \tag{26}$$

with the variational functionals $V_{FV1}[W_1, W_2]$ and $V_{FV2}[W_1, W_2]$ given by:

$$V_{FVi}[W_1, W_2] = \Phi_{FVi}[W_i] - \mu\Psi_{FVi}[W_i] + \Lambda_{FV}[W_1, W_2] \tag{27}$$

Functionals $\Phi_{FVi}(W_i)$, $\Psi_{FVi}[W_i]$ and $\Lambda_{FV}[W_1, W_2]$ in Eq. (27) are given by:

$$\Phi_{FVi}[W_i] = \frac{1}{2}\int_0^L \begin{bmatrix} E_i I_i\left(\frac{d^2 W_i}{dx^2}\right)^2 - P_i\left(\frac{dW_i}{dx}\right)^2 \\ -\omega^2\rho_i A_i W_i^2 + \eta A_i H_x^2\left(\frac{dW_i}{dx}\right)^2 \end{bmatrix} dx \tag{28}$$

$$\Psi_{FVi}[W_i] = \frac{1}{2}\int_0^L \begin{bmatrix} P_i\left(\frac{d^2 W_i}{dx^2}\right)^2 + \omega^2\rho_i A_i\left(\frac{dW_i}{dx}\right)^2 \\ -\eta A_i H_x^2\left(\frac{d^2 W_i}{dx^2}\right)^2 \end{bmatrix} dx \tag{29}$$

$$\Lambda_{FV}[W_1, W_2] = \Lambda_{FV1}[W_1, W_2] + \Lambda_{FV2}[W_1, W_2] \tag{30}$$

where,

$$\Lambda_{FV1}[W_1, W_2] = \frac{k_0}{2}\int_0^L \left[(W_1 - W_2)^2 + \mu\left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right)^2\right] dx \tag{31}$$

$$\Lambda_{FV2}[W_1, W_2] = \frac{G_0}{2}\int_0^L \begin{bmatrix} \left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right)^2 \\ +\mu\left(\frac{d^2 W_1}{dx^2} - \frac{d^2 W_2}{dx^2}\right)^2 \end{bmatrix} dx \tag{32}$$

5.1 Rayleigh quotient for frequency

Next Rayleigh quotient is given for the freely vibrating double beam system. For this purpose, we introduce the functionals $Y_{FVi}(W_i)$ and $M_{FVi}(W_i)$ given by:

$$Y_{FVi}(W_i) = (P_i + \eta A_i H_x^2)\int_0^L \left[\left(\frac{dW_i}{dx}\right)^2 + \mu\left(\frac{d^2 W_i}{dx^2}\right)^2\right] dx \tag{33}$$

$$M_{FVi}(W_i) = \int_0^L E_i I_i\left(\frac{d^2 W_i}{dx^2}\right)^2 dx \tag{34}$$

Using the variational principle given in Eq. (26), Rayleigh quotient for the vibration frequency ω can now be formulated and is given by the expression:

$$\omega^2 = \frac{\sum_{i=1}^{2}[M_{FVi}(W_i) - Y_{FVi}(W_i)] + \Lambda_{FV}[W_1, W_2]}{\sum_{i=1}^{2}\int_0^L \rho_i A_i\left[{W_i}^2 - \mu\left(\frac{dW_i}{dx}\right)^2\right] dx} \tag{35}$$

Frequency ω is computed by minimizing Eq. (35) with respect to W_i. In Eq. (35), $M_{FVi}(W_i)$ is defined by Eq. (34), $Y_{FVi}(W_i)$ by Eq. (33) and $\Lambda_{FV}[W_1, W_2]$ by Eq. (30). It is observed that the effect of the compressive forces P_i and the magnetic field $\eta A_i H_x^2$ is to reduce the frequency. This is due to the fact that the functional $Y_{FVi}(W_i)$ given by Eq. (33) is a positive definite functional and appears with a negative sign in the numerator of the frequency expression (35). The nonlocal effect appears in the denominator of Eq. (35) with a minus sign and in the numerator with a negative and a positive sign (in $\Lambda_{FV}[W_1, W_2]$) and as such its effect on the frequency depends on the relative magnitudes of the problem parameters.

5.2 Rayleigh quotient for buckling load

Next, Rayleigh quotient for buckling load is formulated. For this purpose, we introduce the functionals:

$$Z_{FVi}(W_i) = \int_0^L \rho_i A_i \left[{W_i}^2 - \mu\left(\frac{dW_i}{dx}\right)^2\right] dx \tag{36}$$

$$B_{FVi}(W_i) = \int_0^L \left[\left(\frac{dW_i}{dx}\right)^2 + \mu\left(\frac{d^2W_i}{dx^2}\right)^2\right] dx \tag{37}$$

Rayleigh quotient for the buckling load for the case $P_1 = P_2 = P$ can be expressed as:

$$P = \frac{\sum_{i=1}^{2}\begin{bmatrix} M_{FVi}(W_i) - \omega^2 Z_{FVi}(W_i) \\ -\eta H_x^2 A_i B_{FVi}(W_i) \end{bmatrix} + \Lambda_{FV}(W_1, W_2)}{\sum_{i=1}^{2} B_{FVi}(W_i)} \tag{38}$$

where, $Z_{FVi}(W_i)$ is given by Eq. (36) and $B_{FVi}(W_i)$ by Eq. (37). The buckling load P is computed by minimizing Eq. (38) with respect to W_i. It is observed that the effect of the magnetic field H_x on the buckling load is to reduce the buckling load. This is due to the fact that the functional $B_{FVi}(W_i)$ given by Eq. (37) is a positive definite functional and appears with a negative sign in the numerator of the buckling expression (38). The effect of the nonlocal parameter η on the buckling load cannot be deduced from the buckling expression (38) and can be positive or negative depending on the relative values of various quantities.

6. BOUNDARY CONDITIONS

In this section, natural and geometric boundary conditions are derived using the variational formulation (Eq. (25)) of the freely vibrating double beam system. The first variations of $V_{FV}(W_1, W_2)$ with respect to W_i, denoted as $\delta_{W_i}V_{FV}$, can be obtained by integration by parts. We note that:

$$\delta_{w_1}V_{FV} = \delta_{w_1}V_{FV1} + \delta_{w_1}V_{FV2} = \int_0^L D_1(W_1, W_2)\, \delta W_1\, dx + \partial\Omega_1(0, L) \tag{39a}$$

$$\delta_{w_2}V_{FV} = \delta_{w_2}V_{FV1} + \delta_{w_2}V_{FV2} = \int_0^L D_2(W_1, W_2)\, \delta W_2\, dx + \partial\Omega_2(0, L) \tag{39b}$$

where

$$D_i(W_1, W_2) = L_i(W_i) - \mu N_i(W_i) + (-1)^{i+1}K(W_1, W_2) \tag{40}$$

In Eqs. (38)-(39), $\partial\Omega_i(0, L)$ denotes the boundary terms with $\delta_{W_i}V_{FVi}(W_1, W_2)$ given by:

$$\delta_{W_i}V_{FVi}(W_1, W_2) = \int_0^L \begin{bmatrix} \frac{\partial V_{FVi}}{\partial W_i} \frac{d}{dx}\left(\frac{\partial V_{FVi}}{\partial W_{ix}}\right) \\ + \frac{d^2}{dx^2}\left(\frac{\partial V_{FVi}}{\partial W_{ixx}}\right) \end{bmatrix} \delta W_i\, dx + \partial\Omega_i(0, L) \tag{41}$$

and $\partial\Omega_i(0, L)$ is defined as:

$$\partial\Omega_i(0, L) = (Q_i\, \delta W_i)|_{x=0}^{x=L} + (M_i\, \delta W_{ix})|_{x=0}^{x=L} \tag{42}$$

where,

$$Q_i = -(P_i - \eta A_i H_x^2 + \mu\omega^2\rho_i A_i)\frac{dW_i}{dx} + \mu k_0\left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right) + \mu G_0\left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right) \tag{43}$$

$$M_i = E_i I_i + \mu(P_i - \eta A_i H_x^2)\frac{d^2W_i}{dx^2} + \mu G_0\left(\frac{d^2W_1}{dx^2} - \frac{d^2W_2}{dx^2}\right) \tag{44}$$

Here Q_i is the shear force and M_i is the moment expression. Thus, the boundary conditions at x=0 and x=L are given by:

$$W_i \text{ or } Q_i(x) = -(P_i - \eta A_i H_x^2 + \mu\omega^2\rho_i A_i)\frac{dW_i}{dx} + \mu k_0\left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right) - \mu G_0\left(\frac{dW_1}{dx} - \frac{dW_2}{dx}\right) \text{ specified} \tag{45}$$

$$W_{ix} \text{ or } M_i(x) = E_i I_i + \mu(P_i - \eta A_i H_x^2)\frac{d^2W_i}{dx^2} + G_0\left(\frac{d^2W_1}{dx^2} - \frac{d^2W_2}{dx^2}\right) \text{ specified} \tag{46}$$

with i=1, 2. It is observed that due to Pasternak interlayer between the nanotubes, the boundary conditions are coupled if the shear force and/or moment is specified at x=0 or x=L as given by Eqs. (45)-(46).

Derivations of the variational expressions (26)-(29) and the boundary conditions (45), (46) are given next. We note that:

$$\int_0^L E_i I_i \frac{d^4W_i}{dx^4}\delta W_i\, dx = B_{1iFV}(W_i, \delta W_i) + \delta\left[\frac{1}{2}\int_0^L E_i I_i \left(\frac{d^2W_i}{dx^2}\right)^2 dx\right] \tag{47}$$

where,

$$B_{1FVi}(W_i, \delta W_i) = \left(E_i I_i \frac{d^3W_i}{dx^3}\delta W_i\right)\Big|_{x=0}^{x=L} - \left(E_i I_i \frac{d^2W_i}{dx^2}\delta\left(\frac{dW_i}{dx}\right)\right)\Big|_{x=0}^{x=L} \tag{48}$$

Similarly,

$$\int_0^L (P_i - \eta A_i H_x^2)\frac{d^2W_i}{dx^2}\delta W_i\, dx = B_{2iFV}(W_i, \delta W_i) - \delta\left[\frac{1}{2}\int_0^L (P_i - \eta A_i H_x^2)\left(\frac{dW_i}{dx}\right)^2 dx\right] \tag{49}$$

where,

$$B_{2FVi}(W_i, \delta W_i) = \left((P_i - \eta A_i H_x^2)\frac{dW_i}{dx}\delta W_i\right)\Big|_{x=0}^{x=L} \tag{50}$$

Eqs. (45)-(47) and (49) indicate the variational expressions in Eqs. (34)-(35). Similarly, Eqs. (46), (48), and (50) have the boundary terms which appear in the boundary conditions shown in Eqs. (43)-(44).

7. CONCLUSIONS

A system of two partial differential equations govern the vibrations of a system of double single-walled carbon nanotubes with the connection between the nanotubes modelled as an elastic interlayer. The double nanotube system is subject to a longitudinal magnetic field and compressive forces. In the present study, the constitutive equations are based on nonlocal Euler-Bernouilli beam theory and the interlayer between the nanotubes is defined in the form of a Winkler-Pasternak layer with the expressions involving the displacements (Winkler) and second derivatives of displacements (Pasternak). As the constitutive relations are based on nonlocal elasticity, the formulation leads to fourth order derivatives describing the interlayer. The variational formulation of the problem is obtained for the coupled differential equations with the system subject to forced vibrations. For use in the approximate and variational methods of solutions of the problem, vibration frequency and the buckling load are expressed in the form of Rayleigh quotients. A study of the Rayleigh quotients indicates that the effect of the magnetic field is to reduce the frequency and the buckling load. An important part of the present study is the derivation of variationally consistent boundary conditions. Natural boundary conditions are obtained involving the moment and shear force expressions. It is observed that the presence of a Pasternak interlayer between the nanotubes leads to coupled boundary conditions. The results presented in the study can be used in a number of approximate and numerical methods of solution, especially in the presence of different boundary conditions.

It is noted that variational formulation of a problem defined by differential equations provide the weak formulation of the problem and the approximations based on this formulation are the weak solutions. However, weak formulations can provide accurate solutions of the differential equations in the absence of an analytical solution as noted in Nicolescu and Bobe [71]. Future work on the present study could involve obtaining the approximate solutions of the nanotube system subject to a combination of clamped, free, simply supported and rotational/torsional boundary conditions using the variational formulation of the present study. This would require choosing suitable trial functions for the approximate solution of the specific problem based on the applicable boundary conditions. Another direction in further work can involve the study of the second variation of the variational functional and determining the conditions under which it is positive-definite indicating a minimum. Moreover, present formulation is based on Euler-Bernoulli beam theory and further work can extend the results formulation based on Timoshenko beam theory in order to take the shear effects into consideration.

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