Numerical Finite Difference Scheme for a Two-Dimensional Time-Fractional Semilinear Diffusion Equation

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ABSTRACT

Time-fractional partial differential equations are foundational instruments in modeling neuronal dynamics. These equations are formulated by replacing the conventional time derivative of order α, where $0 < \alpha < 1$, in the standard equation with the Caputo fractional derivative. This study introduces the Crank-Nicolson (C.N.) finite difference scheme as a solution method for a two-dimensional, time-fractional Semilinear parabolic equation under Dirichlet boundary conditions. An in-depth investigation into the consistency, stability, and convergence of the proposed scheme is also conducted. To corroborate the theoretical findings, two numerical experiments are carried out. The scheme's efficiency, in terms of absolute errors, order of accuracy, and computational time, is meticulously evaluated and discussed. The results demonstrate that the proposed scheme, while being conditionally stable, can be utilized effectively with a high rate of convergence to compute numerical solutions for the problem at hand.

1. INTRODUCTION

Fractional calculus, a branch of mathematics concerned with the properties of derivatives and integrals of non-integer orders, has been an area of interest since the inception of classical calculus. This mathematical field specializes in the resolution of time-dependent fractional differential equations, which entail fractional derivatives.

Regarded as a pivotal tool for the exploration of dynamical systems, fractional calculus is valued for its non-local operators that encapsulate the historical progression of dynamics. The usage of fractional calculus and fractional processes has become a favored approach when grappling with the unique properties of long-term memory effects found in multiple domains of applied sciences. These fields range from finance and economic dynamics, biological systems and bioinformatics, nonlinear waves and acoustics, to image and signal processing, transportation systems, geosciences, astronomy, and cosmology.

The development of fractional calculus has been incrementally built upon by an array of researchers including, but not limited to, Heaviside, Lagrange Riemann, Liouville, Grïunwald, Euler, Fourier, and Abel [1]. The contemporary utilization of fractional calculus owes its popularity to the versatility of the differintegral operator, which amalgamates both integer-order derivatives and integrals as special cases. The fractional integral, for instance, may be employed to describe the cumulation of a quantity when the order of integration is undetermined, and can be inferred as a parameter of a regression model, as elucidated by Podlubny [2] and Kisela [3]. The fractional derivative, on the other hand, is frequently used to depict damping. Other applications span control theory of dynamical systems, optics and signal processing, fluid flow, diffusive transport, probability and

statistics, viscoelasticity, electrical networks, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, and rheology.

Over the past decades, several analytical and numerical methods for solving fractional differential equations have been brought to bear. Analytical methods encompass the Fourier transformation method, the Laplace transformation method, and the green function method [4-6]. However, most fractional differential equations resist analytical solutions. Hence, the development of numerical schemes for these equations is imperative. Effective methods such as the finite difference method, the spectral method, and the finite element method have been utilized to solve time (space) fractional differential equations [7-9].

The numerical solutions of linear types of time (space)-fractional equations have been the focal point of many authors [9-13], while Semilinear or nonlinear types have received less attention [14-17]. Nevertheless, research on these latter types is still in its nascent stages.

Since, most of the mathematical models, that describe real would phenomena, involving non-linear fractional partial differential equations, and most of previous studies have focused on linear types, the aim of this work is to suggest a high order efficient numerical technique for a two-dimensional semi-linear time fractional equation with Dirichlet boundary conditions. Namely, we propose the linearly implicit Crank-Nicolson finite difference scheme, and we show that the proposed scheme is consistent, conditionally stable and convergent with the help of mathematical induction. Moreover, we prove that the order of accuracy takes the form: $O(k^{2-\alpha} + h^2)$, where α is the fractional order of the time-derivative. In addition, the obtained numerical results of the studied test examples show that the proposed scheme can be used efficiently to compute the numerical solutions of the

governed problem.

The rest of this paper is organized as follows: In section two, we state the mathematical formulation of the time fractional diffusion equation and present some literature review regarding the studied problem. In section three the derivation, consistency, stability and convergent of the finite difference scheme are considered in section four. Finally, some conclusions are given in the last section.

2. TIME-FRACTIONAL DIFFUSION EQUATION

Since the last decades, a lot of interest has been shown in solving the two-dimensional linear time fractional diffusion equation associated with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \\ u(x,0,t) = a(x,t), u(x,1,t) = b(x,t), \\ u(0,y,t) = c(y,t), u(1,y,t) = d(y,t), \\ u(x,y,0) = u_0(x,y), \\ \text{for } \alpha \in (0,1), \ 0 < x < 1, 0 < y < 1, 0 < t < T \end{cases} \tag{1}$$

where, $a, b, c, d, u_0 \in C(R^2)$, $f \in C(R^3)$.

In fact, problem (1) has many applications, such as it is used to describe transport processes with long memory, where the rate of diffusion is inconsistent with the classical Brownian motion model [9].

There are various finite difference schemes such as explicit and implicit Euler and Crank - Nicolson schemes have been developed to solve this equation. For instance, Zhuang and Liu [18] solved problem (1) with the help of the implicit difference approximation technique. They used Caputo fractional definition for approximating the time fractional derivative. They proved that the implicit difference scheme was unconditionally stable and convergent with the help of mathematical induction. Later, Chen et al. [19], established the implicit and the explicit finite difference schemes for providing solutions for the problem (1). They discretized the time derivatives with the help of the Riemann-Liouville fractional derivative and showed the connection between the Riemann-Liouville and the Grünwald-Letnikov fractional derivatives. Thereafter, the authors could apply the Riemann-Liouville definition for problem formulations and use the Grünwald-Letnikov definition for the derivation of the numerical solutions. The Fourier analysis technique is used for determining the stability and the convergence. Furthermore, they used the multivariate extrapolation method for improving the method accuracy.

Zhang and Sun [20] presented two new Alternating Direction Implicit (ADI) schemes, which were based on the L_1 approximation and the backward Euler technique for solving problem (1). These ADI schemes have been constructed after adding the two small different terms that are different from the general ADI technique. They proved the solvability, H_1 norm convergence and the unconditional stability. Cui [21] investigated the high-order compact finite difference process with the operator splitting method (ADI approach) for solving problem (1) by the approximation of the 2nd order spatial derivatives using the compact finite difference and applying the Grünwald-Letnikov discretisation scheme for time fractional derivatives. They proved that the method was unconditionally stable using the Fourier approach.

Similar to their earlier study, Cui [22] suggested the compact ADI scheme for providing solutions to the problem (1). They used the Caputo derivative for approximating the time fractional derivative instead of using the Grünwald-Letnikov as their earlier study. They approximated the second order derivatives with regards to the space variables by applying the compact finite difference. The Fourier analysis technique was used for proving the convergence for the compact finite difference.

Later, Balasim and Ali [11] have proposed the group iterative methods to compute the numerical solutions of problem (1). Balasim [23] has investigated the convergence and stability of the group iterative schemes.

In order to generalize some of previous results for nonlinear time fractional diffusion equations, in this work, we propose a numerical technique for a two-dimensional time fractional reaction diffusion equations of a Semilinear type associated with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t,u), \\ u(x,0,t) = a(x,t), u(x,1,t) = b(x,t), \\ u(0,y,t) = c(y,t), u(1,y,t) = d(y,t), \\ u(x,y,0) = u_0(x,y), \\ \text{for } 0 < x < 1, 0 < y < 1, 0 < t < T \end{cases} \tag{2}$$

where, $a, b, c, d, u_0 \in C(R \times R), 0 \le u_0,$
$f \in C^1([0,1] \times [0,1] \times (0,T) \times [0,\infty))$ and satisfies Lipshtiz condition with respect to u:

$$\begin{aligned} |f(x,y,t,u_1) - f(x,y,t,u_2)| &\le L|u_1 - u_2|, \\ \forall u_1, u_2, L &> 0 \end{aligned} \tag{3}$$

And $\alpha \in (0,1)$ is the order of the time fractional derivative in Caputo sense [12, 15], which takes the form:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,y,\xi)}{\partial \xi} \frac{\partial \xi}{(t-\xi)^\alpha} \tag{4}$$

The existence, uniqueness and stability of problem (2) can be guaranteed by some references [24, 25]. Throughout this paper, we assume that the solution of problem (2) is positive for nonzero time values.

3. THE LINEARLY IMPLICIT CRANK-NICOLSON METHOD (CNM)

In this section, we propose a numerical approximation to problem (2). Namely, the linearly Crank-Nicolson method (CNM). Firstly, we derive its formula, then we investigate its consistency, stability and convergence.

In this segment, the grid dimensions in relation to space and time for the positive integers I and N are respectively represented by $h = \frac{1}{I}$ and $k = \frac{T}{N}$. The grid point in the space interval $[0,1]$ is denoted $x_i = ih$, $y_j = jh$, $i,j = 0,1,2,\dots,I$ and the grid points for time are designated $t_n = nk, n = 0,1,2 \dots N$. In addition, we denote $u_i^n = u(x_i, y_j, t_n)$.

3.1 Crank Nicolson formula

The goal of this section is to derive Crank-Nicolson scheme for problem (2), which is one of the most popular methods in

practice. Moreover, it has high order of convergence in both space and time.

In order to approximate Eq. (2) and Eq. (4) at the mush point $\left(x_i, y_j, t_{n+\frac{1}{2}}\right)$, we use the following approximations [10, 13]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}\Big|_i^{n+\frac{1}{2}} = \left[\widetilde{w}_1 u_{i,j}^n + \sum_{s=1}^{n-1}(\widetilde{w}_{n-s+1} - \widetilde{w}_{n-s}) u_{i,j}^s - \widetilde{w}_n u_{i,j}^0 + \sigma\left(\frac{u_{i,j}^{n+1} + u_{i,j}^n}{2^{1-\alpha}}\right)\right] + O(k^{(2-\alpha)}) \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+\frac{1}{2}} = \left[\left(\frac{u_{i+1,j}^{n+1} + -2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{2h^2}\right)\right] + \left[\left(\frac{u_{i+1,j}^n + -2u_{i,j}^n + u_{i-1,j}^n}{2h^2}\right)\right] + O(h^2) \tag{6}$$

$$\frac{\partial^2 u}{\partial y^2}\Big|_i^{n+\frac{1}{2}} = \left[\left(\frac{u_{i,j+1}^{n+1} + -2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{2h^2}\right)\right] + \left[\left(\frac{u_{i,j+1}^n + -2u_{i,j}^n + u_{i,j-1}^n}{2h^2}\right)\right] + O(h^2)$$

$$\sigma = \frac{1}{k^\alpha \Gamma(2-\alpha)}, \ \widetilde{w}_s = \sigma\left[\left(s+\frac{1}{2}\right)^{(1-\alpha)} - \left(s - \frac{1}{2}\right)^{(1-\alpha)}\right] \tag{7}$$

The nonlinear term can be approximated using Taylor expansion [16]:

$$f\left(x_i, y_j, t_{n+\frac{1}{2}}, u\left(x_i, y_j, t_{n+\frac{1}{2}}\right)\right) = f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u(x_i, y_j, t_n) - \frac{1}{2}u(x_i, y_j, t_{n-1})\right) + O(k^2)$$

Thus

$$f\left(x_i, y_j, t_{n+\frac{1}{2}}, u\left(x_i, y_j, t_{n+\frac{1}{2}}\right)\right) = f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right) + O(k^2) \tag{8}$$

By substituting the above forms (5)-(8) in problem (2), we can get the Crank – Nicolson approximated formula as follows:

$$\left[\widetilde{w}_1 u_{i,j}^n + \sum_{s=1}^{n-1}(\widetilde{w}_{n-s+1} - \widetilde{w}_{n-s}) u_{i,j}^s - \widetilde{w}_n u_{i,j}^0 + \sigma\left(\frac{u_{i,j}^{n+1} - u_{i,j}^n}{2^{1-\alpha}}\right)\right]$$
$$= \frac{1}{2h^2}\left[(u_{i+1,j}^{n+1} + -2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}) + (u_{i+1,j}^n + -2u_{i,j}^n + u_{i-1,j}^n)\right] + \frac{1}{2h^2}\left[(u_{i,j+1}^{n+1} + -2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}) + (u_{i,j+1}^n + -2u_{i,j}^n + u_{i,j-1}^n)\right]$$
$$+ f(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}).$$

Multiplying both sides of the last equation by $\delta = k^\alpha \Gamma(2 - \alpha)2^{1-\alpha}$, yields that:

$$\left[2^{1-\alpha}w_1 u^n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s+1} - w_{n-s}) u_{i,j}^s - 2^{1-\alpha}w_n u_{i,j}^0 + u_{i,j}^{n+1} - u_{i,j}^n\right] = \frac{\delta}{2h^2}\left[(u_{i+1,j}^{n+1} + -2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}) + (u_{i+1,j}^n + -2u_{i,j}^n + u_{i-1,j}^n)\right] + \frac{\delta}{2h^2}\left[(u_{i,j+1}^{n+1} + -2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}) + (u_{i,j+1}^n + -2u_{i,j}^n + u_{i,j-1}^n)\right]$$
$$+ \delta f(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}).$$

So,

$$u_{i,j}^{n+1} - \frac{\delta}{2h^2}\left(u_{i+1,j}^{n+1} + + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) + \frac{2\delta}{h^2} u_{i,j}^{n+1}$$
$$= \frac{\delta}{2h^2}\left(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n\right) - \frac{2\delta}{h^2} u_{i,j}^n + u_{i,j}^n - 2^{1-\alpha}w_1 u_{i,j}^n$$
$$+ 2^{1-\alpha}\sum_{s=1}^n (w_{n-s} - w_{n+1-s})u_{i,j}^s + 2^{1-\alpha}w_n u_{i,j}^0$$
$$+ \delta f(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}).$$

It follows

$$(1 + 2r) u_{i,j}^{n+1} - \frac{r}{2}\left(u_{i+1,j}^{n+1} + + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right) = \frac{r}{2}\left(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n\right)$$
$$+(1 - 2r - 2^{1-\alpha}w_1)u_{i,j}^n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n+1-s})u_{i,j}^s + 2^{1-\alpha}w_n u_{i,j}^0$$
$$+ \delta f(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}).$$

We obtain

$$(1 + 2r)u_{i,j}^{n+1} - \frac{r}{2}\left[u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right]$$
$$= (1 - 2^{1-\alpha}w_1 - 2r)u_{i,j}^n$$
$$+ \frac{r}{2}\left[u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n\right]$$
$$+ 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})u_{i,j}^s$$
$$+ 2^{1-\alpha}w_n u_{i,j}^0$$
$$+ \delta f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{ij}^{n-1}\right) \tag{9}$$

where, $\delta = k^\alpha \Gamma(2 - \alpha)2^{1-\alpha}$, $r = \frac{\delta}{h^2}$, $w_s = \left[\left(s+\frac{1}{2}\right)^{(1-\alpha)} - \left(s - \frac{1}{2}\right)^{(1-\alpha)}\right]$

For $n=0$, CN formula (9) becomes as follows:

$$(1 + 2r)u_{i,j}^1 - \frac{r}{2}\left[u_{i+1,j}^1 + u_{i-1,j}^1 + u_{i,j+1}^1 + u_{i,j-1}^1\right] = (1 - 2r)u_{i,j}^0 + \frac{r}{2}\left[u_{i+1,j}^0 + u_{i-1,j}^0 + u_{i,j+1}^0 + u_{i,j-1}^0\right] + \delta f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^0\right).$$

Lemma 3.1 [13]
For $s = 0, 1, 2\ldots, n$
- $w_s > 0$
- $w_{s+1} < w_s$, $w_{s+1}^{-1} > w_s^{-1}$
- $\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) + w_n = w_1$.

The Crank – Nicolson difference formula (5) can be written in matrix form as follows:

$$\begin{cases} A\,U^1 = BU^0 + \delta F^0 + Z^1 + Z^0 \\ A\,U^{n+1} = BU^n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})U^s \\ \qquad + 2^{1-\alpha}w_n U^0 + \delta F^n + Z^{n+1} + Z^n \end{cases} \quad (10)$$

$$A = \begin{bmatrix} A^* & -\frac{r}{2}I & 0 & \cdots & 0 \\ -\frac{r}{2}I & A^* & -\frac{r}{2}I & \cdots & 0 \\ & \ddots & & & \\ & & \ddots & & \\ 0 & \cdots & 0 & -\frac{r}{2}I & A^* \end{bmatrix}_{(I-1)^2 \times (I-1)^2}$$

$$B = \begin{bmatrix} B^* & \frac{r}{2}I & 0 & \cdots & 0 \\ \frac{r}{2}I & B^* & \frac{r}{2}I & \cdots & 0 \\ & \ddots & & & \\ & & \ddots & & \\ 0 & \cdots & 0 & \frac{r}{2}I & B^* \end{bmatrix}_{(I-1)^2 \times (I-1)^2}$$

$$A^* = \begin{bmatrix} (1+2r) & -\frac{r}{2} & 0 & \cdots & 0 \\ -\frac{r}{2} & (1+2r) & -\frac{r}{2} & \cdots & 0 \\ & \ddots & & & \\ & & \ddots & & \\ 0 & \cdots & 0 & -\frac{r}{2} & (1+2r) \end{bmatrix}_{(I-1)\times(I-1)}$$

$$B^* = \begin{bmatrix} (1-2r-2^{1-\alpha}w_1) & \frac{r}{2} & 0 & \cdots & 0 \\ \frac{r}{2} & (1-2r-2^{1-\alpha}w_1) & \frac{r}{2} & \cdots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & \cdots & 0 & \frac{r}{2} & (1-2r-2^{1-\alpha}w_1) \end{bmatrix}_{(I-1)\times(I-1)}$$

$$U^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{I-1}^n \end{bmatrix}, F^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \end{bmatrix},$$

$$u_i^n = \begin{bmatrix} u_{i,1}^n & u_{i,2}^n & \cdots & u_{i,I-1}^n \end{bmatrix}^T, \quad f_i^n$$
$$= \begin{bmatrix} f_{i,1}^n & f_{i,2}^n & \cdots & f_{i,I-1}^n \end{bmatrix}^T$$
$$f_{i,j}^n = f\left(x_i, y_j, t_{n\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{ij}^{n-1}\right).$$

$$Z^n = \frac{r}{2}\big[u_{0,1}^n + u_{1,0}^n, u_{0,2}^n, \ldots, u_{0,I-2}^n, u_{0,I-1}^n$$
$$+ u_{1,I}^n; u_{2,0}^n, 0, \ldots, 0, u_{2,I}^n; \ldots; u_{I,1}^n$$
$$+ u_{I-1,0}^n, u_{I,2}^n, \ldots, u_{I,I-2}^n, u_{I,I-1}^n + u_{I-1,I}^n\big]^T$$

Theorem 3.1 At each time level, (n+1), the linear system (10) is uniquely salable.

Proof: Since A is diagonally dominant with positive real diagonal entries, then A is positive definite and nonsingular [26]. Hence, the linear system (10) is uniquely solvable.

3.2 Stability analysis for C.N. method

Suppose that $\tilde{u}_i^n$ is the approximate solution of Eq. (2). As in studies [18, 27], we set $e_{i,j}^n = \tilde{u}_{i,j}^n - u_{i,j}^n$, $i,j = 0,1,2,\ldots,I-1$, $n = 0,1,2,\ldots,N$.
Define $\|E^n\| = Max_{\substack{1\le i \le I-1 \\ 1\le j \le I-1}} |e_{i,j}^n|$
Clearly, $e_{i,j}^n$ satisfies (9):

For $n=0$,

$$(1+2r)e_{i,j}^1 - \frac{r}{2}\left[e_{i+1,j}^1 + e_{i-1,j}^1 + e_{i,j+1}^1 + e_{i,j-1}^1\right]$$
$$= (1-2r)e_{i,j}^0 - \frac{r}{2}\left[e_{i+1,j}^0 + e_{i-1,j}^0 + e_{i,j+1}^0 + e_{i,j-1}^0\right] + \delta\left[f\left(x_i, y_j, t_{\frac{1}{2}}, \frac{3}{2}\tilde{u}_{i,j}^0\right) - f\left(x_i, y_j, t_{\frac{1}{2}}, \frac{3}{2}u_{i,j}^0\right)\right]. \quad (11)$$

For $n>0$,

$$(1+2r)e_{i,j}^{n+1} - \frac{r}{2}\left[e_{i+1,j}^{n+1} + e_{i-1,j}^{n+1} + e_{i,j+1}^{n+1} + e_{i,j-1}^{n+1}\right]$$
$$= (1 - 2^{1-\alpha}w_1 - 2r)e_{i,j}^n + \frac{r}{2}\left[e_{i+1,j}^n + e_{i-1,j}^n + e_{i,j+1}^n + e_{i,j-1}^n\right] + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})e_{i,j}^n + 2^{1-\alpha}w_n e_{i,j}^0$$
$$+ \delta\left[f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}\tilde{u}_{i,j}^n - \frac{1}{2}\tilde{u}_{i,j}^{n-1}\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right)\right]. \quad (12)$$

Based on (10), the above error formula can be written in matrix form as follows:

$$\begin{cases} AE^1 = BE^0 + \delta G^0 \\ AE^{n+1} = BE^n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})E^s \\ \qquad + 2^{1-\alpha}w_n E^0 + \delta G^n \end{cases}$$

E^0 is given,

$$E^n = \begin{bmatrix} E_1^n \\ E_2^n \\ \vdots \\ E_{I-1}^n \end{bmatrix}, G^n = \begin{bmatrix} G_1^n \\ G_2^n \\ \vdots \\ G_{I-1}^n \end{bmatrix},$$

$$E_i^n = \begin{bmatrix} e_{i,1}^n & e_{i,2}^n & \cdots & e_{i,I-1}^n \end{bmatrix}^T, \quad G_i^n$$
$$= \begin{bmatrix} g_{i,1}^n & g_{i,2}^n & \cdots & g_{i,I-1}^n \end{bmatrix}^T$$
$$g_{i,j}^n = f\left(x_i, y_j, t_{n\frac{1}{2}}, \frac{3}{2}\tilde{u}_{i,j}^n - \frac{1}{2}\tilde{u}_{ij}^{n-1}\right) - f\left(x_i, y_j, t_{n\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{ij}^{n-1}\right).$$
$$i,j = 0,1,2,\ldots,I-1, n = 0,1,2,\ldots,N$$

Definition 3.1 [14] For any arbitrary initial rounding error E^0, the difference approximation $AU^{n+1} = BU^n + b^n$ is stable, if there exists $C > 0$, independent on h, k such that $\|E^n\| \le C\|E^0\|$, or $\|(A^{-1}B)^n\| \le C$, $\forall n$.

Theorem 3.2 The C.N. finite difference approximation is stable, if $(1 - 2^{1-\alpha}w_1 - 2r) \ge 0$.
Proof: To prove this theorem, we use the Mathematical induction.
For $n=0$, set $\|E^1\|_\infty = Max_{\substack{1\le i \le I-1 \\ 1\le j \le I-1}} |e_{i,j}^1| = |e_{p,q}^1|$, we have

$$|e_{p,q}^1| = (1+2r)|e_{p,q}^1| - \frac{r}{2}\left(4|e_{p,q}^1|\right)$$
$$\le (1+2r)|e_{p,q}^1| - \frac{r}{2}\left(|e_{p+1,q}^1| + |e_{p-1,q}^1| + |e_{p,q+1}^1| + |e_{p,q-1}^1|\right)$$
$$\le \left|(1+2r)e_{p,q}^1 - \frac{r}{2}\left(e_{p+1,q}^1 + e_{p-1,q}^1 + e_{p,q+1}^1 + e_{p,q-1}^1\right)\right| \quad (13)$$

From (11) and (13), we obtain

$$\left|e_{p,q}^1\right| \le \left|(1-2r)e_{p,q}^0 + \frac{r}{2}\left(e_{p+1,q}^0 + e_{p-1,q}^0 + e_{p,q+1}^0 + e_{p,q-1}^0\right)\right| + \delta\left|f\left(x_p, y_q, t_{\frac{1}{2}}, \frac{3}{2}\tilde{u}_{p,q}^0\right) - f\left(x_p, y_q, t_{\frac{1}{2}}, \frac{3}{2}u_{p,q}^0\right)\right|$$

Since satisfies Lipchitz condition (3), we obtain

$$ep, ql \le 1-2rE0\infty+2rE0\infty\left|e_{p,q}^1\right| \le (1-2r)\|E^0\|_\infty + 2r\|E^0\|_\infty$$
$$+\delta L\left|\frac{3}{2}\left(\tilde{u}_{p,q}^0 - u_{p,q}^0\right)\right|.$$
$$\le \|E^0\|_\infty + \frac{3}{2}\delta L\|E^0\|_\infty = \left(1 + \frac{3}{2}\delta L\right)\|E^0\|_\infty$$
$$\le (1 + 2\delta L)\|E^0\|_\infty.$$

Thus $\|E^1\|_\infty \le (1 + 2\delta L)\|E^0\|_\infty$

Now, supposes that $\|E^s\|_\infty \le C\ \|E^0\|_\infty$, where $C=(1+2\delta L)$, $s=0, 1, 2, \ldots, n.$, for $n+$, let $\left|e_{p,q}^{n+1}\right| = Max_{\substack{1 \le i \le I-1 \\ 1 \le j \le I-1}}\left|e_{i,j}^{n+1}\right|$, we have

$$\left|e_{p,q}^{n+1}\right| = (1+2r)\left|e_{p,q}^{n+1}\right| - \frac{r}{2}\left[4\left|e_{p,q}^{n+1}\right|\right].$$
$$\le (1+2r)\left|e_{p,q}^{n+1}\right| - \frac{r}{2}\left(\left|e_{p+1,q}^{n+1}\right| + \left|e_{p-1,q}^{n+1}\right| + \left|e_{p,q+1}^{n+1}\right| + \left|e_{p,q-1}^{n+1}\right|\right). \qquad (14)$$
$$\le \left|(1+2r)\,e_{p,q}^{n+1} - \frac{r}{2}\left(e_{p+1,q}^{n+1} + e_{p-1,q}^{n+1} + e_{p,q+1}^{n+1} + e_{p,q-1}^{n+1}\right)\right|.$$

From (12) and (14), it follows that

$$\left|e_{p,q}^{n+1}\right| \le \left|(1 - 2^{1-\alpha}w_1 - 2r)e_{p,q}^n + \frac{r}{2}\left(e_{p+1,q}^n + e_{p-1,q}^n + e_{p,q+1}^n + e_{p,q-1}^n\right)\right| + \left|2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,e_{p,q}^s\right| + \delta\left|f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}\tilde{u}_{p,q}^n - \frac{1}{2}\tilde{u}_{p,q}^{n-1}\right) - f\left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2}u_{p,q}^n - \frac{1}{2}u_{p,q}^{n-1}\right)\right|.$$

Since satisfies Lipchitz condition (3), we obtain

$$\left|e_{p,q}^{n+1}\right| \le (1 - 2^{1-\alpha}w_1 - 2r)\left|e_{p,q}^n\right| + \frac{r}{2}\left(\left|e_{p+1,q}^n\right| + \left|e_{p-1,q}^n\right| + \left|e_{p,q+1}^n\right| + \left|e_{p,q-1}^n\right|\right) + 2^{1-\alpha}w_n\left|e_{p,q}^0\right| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\left|e_{i,j}^s\right| + \delta L\left|\frac{3}{2}\left(\tilde{u}_{p,q}^n - u_{p,q}^n\right) + \frac{1}{2}\left(u_{p,q}^{n-1} - \tilde{u}_{p,q}^{n-1}\right)\right|.$$
$$\le (1 - 2^{1-\alpha}w_1 - 2r)\|E^n\|_\infty + 2r\|E^n\|_\infty + 2^{1-\alpha}w_n\|E^0\|_\infty + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\|E^s\|_\infty + \delta L\left[\frac{3}{2}\left|e_{p,q}^n\right| + \frac{1}{2}\left|e_{p,q}^{n-1}\right|\right].$$
$$\le (1 - 2^{1-\alpha}w_1)\,C^n\|E^n\|_\infty + 2^{1-\alpha}w_n\ \|E^0\| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,C^s\ \|E^0\| + \delta L\left[\frac{3}{2}\ C^n\ \|E^0\| + \frac{1}{2}\ C^{n-1}\ \|E^0\|\right].$$
$$\le (1 - 2^{1-\alpha}w_1)\,C^n\|E^0\| + 2^{1-\alpha}w_n C^n\ \|E^0\| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,C^n\ \|E^0\| + 2\delta L\ C^n\ \|E^0\|$$
$$= [1 - 2^{1-\alpha}w_1 + 2^{1-\alpha}w_n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) + 2\delta L\,]C^n\|E^0\|.$$
$$= (1 + 2\delta L\,)\,C^n\ \|E^0\| = (1 + 2\delta L\,)^{n+1}\ \|E^0\|.$$

Thus
$$\|E^{n+1}\|_\infty \le (1 + 2\delta L\,)^{n+1}\|E^0\|.$$

So, $\|E^{n+1}\| \le (1 + 2^{2-\alpha}k^\alpha\Gamma(2-\alpha)L)^{n+1}\|E^0\|_\infty$
$$\le (1 + 2^{2-\alpha}T^\alpha\Gamma(2-\alpha)L)^{n+1}\|E^0\|_\infty$$

Hence, there exists $C>0$, such that $\|E^{n+1}\| \le C\|E^0\|_\infty$

3.3 Convergence analysis of Crank – Nicolson method

Let $u_{i,j}^n = u(x_i, y_j, t_n)$ be the exact solution of problem (2) at mesh point $(x_i, y_j, t_n), i, j = 0, 1, 2, \ldots, I-1, n = 0, 1, 2, \ldots, N$.

Definition 3.2
$e_{i,j}^n = u(x_i, y_j, t_n) - u_{i,j}^n$, $n = 0, 1, 2, \ldots, N, i, j = 0, 1, 2, \ldots, I-1$

$$E^n = \begin{bmatrix} E_1^n \\ E_2^n \\ \vdots \\ E_{I-1}^n \end{bmatrix}, E^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_i^n = \begin{bmatrix} e_{i,1}^n & e_{i,2}^n & \cdots & e_{i,I-1}^n \end{bmatrix}^T$$
$$i = 0, 1, 2, \ldots, I-1 , n = 0, 1, 2, \ldots, N$$

By substitution $e_{i,j}^n$ in the C.N. Eq. (9), it follows that
For $n=0$,

$$(1+2r)e_{i,j}^1 - \frac{r}{2}\left[e_{i+1,j}^1 + e_{i-1,j}^1 + e_{i,j+1}^1 + e_{i,j-1}^1\right]$$
$$= \delta\left[f\left(x_i, y_j, t_{\frac{1}{2}}, \frac{3}{2}u(x_i, y_j, \frac{3}{2}u(x_i, y_j, t_0)\right) - f\left(x_i, y_j, t_{\frac{1}{2}}, \frac{3}{2}u_{i,j}^0\right)\right] + T_{i,j}^{\frac{1}{2}} \qquad (15)$$

For $n>0$,

$$(1+2r)e_{i,j}^{n+1} - \frac{r}{2}\left[e_{i+1,j}^{n+1} + e_{i-1,j}^{n+1} + e_{i,j+1}^{n+1} + e_{i,j-1}^{n+1}\right]$$
$$= (1 - 2^{1-\alpha}w_1 - 2r)\,e_{i,j}^n + \frac{r}{2}\left[e_{i+1,j}^n + e_{i-1,j}^n + e_{i,j+1}^n + e_{i,j-1}^n\right] + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,e_{i,j}^s + \delta\left[f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u(x_i, y_j, t_n) - \frac{1}{2}u(x_i, y_j, t_{n-1})\right) - f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right)\right] + T_{i,j}^{n+1} \qquad (16)$$

Theorem 3.3 There exists $C>0$ such that:

$$\left|T_{i,j}\right| \le C(k^{2-\alpha} + h^2)$$

Proof: Let $u_{i,j}^n = u\left(x_i, y_j, t_n\right)$ be the exact solution of problem (2).

$$T_{i,j}^{n+\frac{1}{2}} = \left(u_{i,j}^{n+1} - u_{i,j}^n\right) + 2^{1-\alpha}w_1u_{i,j}^n - 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,u_{i,j}^s - 2^{1-\alpha}w_n - \frac{r}{2}\left[u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1} - 4\,u_{i,j}^{n+1}\right] - \frac{r}{2}\left[u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4\,u_{i,j}^n\right] - \delta f\left(x_i, y_j, t_{n+\frac{1}{2}}, \frac{3}{2}u_{i,j}^n - \frac{1}{2}u_{i,j}^{n-1}\right).$$

Thus

$$k^\alpha\Gamma(2-\alpha)\,2^{1-\alpha}\left[u_t^\alpha\left(x_i, y_j, t_{n+\frac{1}{2}}\right) + O(k^{2-\alpha})\right] -$$

$$k^{\alpha} \Gamma(2 - \alpha) \, 2^{1-\alpha} \left[u_{xx}(x_i, y_j, t_{n+1}) + u_{xx}(x_i, y_j, t_n) + O(h^2) \right]$$

$$- k^{\alpha} \Gamma \left[(2 - \alpha) \, 2^{1-\alpha} f \left(x_i, y_j, t_{n+\frac{1}{2}}, u_{i,j}^{n+\frac{1}{2}} \right) + O(k^2) \right]$$

$$= k^{\alpha} \Gamma(2 - \alpha) \, 2^{1-\alpha} [u_t - u_{xx} - u_{yy} - f]_{i,j}^{n+\frac{1}{2}} + k^{\alpha} \Gamma(2 - \alpha) \, 2^{1-\alpha} [O(k^{2-\alpha}) + O(h^2) + O(k^2)]$$

Thus $\left| T_{i,j}^{n+\frac{1}{2}} \right| \leq C(k^{2-\alpha} + h^2)$, C>0.

Remark 3.1 Based on Theorem 3.7, the C.N. formula (9) is consistent.

$$\left| T_{i,j}^{n+\frac{1}{2}} \right| \to 0, \quad h, k \to 0 .$$

Theorem 3.4 If $(1 - 2r - 2^{1-\alpha} w_1) \geq 0$, there exists $C > 0$ such that

$$\left| e_{i,j}^{n+1} \right| \leq C(k^{2-\alpha} + h^2), i, j = 1, 2, \ldots, I - 1 , \quad n = 0, 1, \ldots, N$$

Proof: Define $\|E^n\|_{\infty} = Max_{\substack{1 \leq i \leq I-1 \\ 1 \leq j \leq I-1}} \left| e_{i,j}^n \right|$.

By using mathematical induction, we can prove this theorem as follows:

For n=0, let $\|E^1\|_{\infty} = \left| e_{p,q}^1 \right| = Max_{\substack{1 \leq i \leq I-1 \\ 1 \leq j \leq I-1}} \left| e_{i,j}^1 \right|$.

$$\left| e_{i,j}^1 \right| \leq \left| e_{p,q}^1 \right| = (1 + 2r) \left| e_{p,q}^1 \right| - \frac{r}{2} \left(4 \left| e_{p,q}^1 \right| \right).$$
$$\leq (1 + 2r) \left| e_{p,q}^1 \right| - \frac{r}{2} \left(\left| e_{p+1,q}^1 \right| + \left| e_{p-1,q}^1 \right| + \left| e_{p,q+1}^1 \right| + \left| e_{p,q-1}^1 \right| \right) .$$
$$\leq \left| (1 + 2r) e_{p,q}^1 - \frac{r}{2} \left(e_{p+1,q}^1 + e_{p-1,q}^1 + e_{p,q+1}^1 + e_{p,q-1}^1 \right) \right| \tag{17}$$

From (15) and (17), it follows that

$$\left| e_{i,j}^1 \right| \leq \left| \delta \left[f \left(x_p, y_q, t_{\frac{1}{2}}, \frac{3}{2} u(x_p, y_q, t_0) \right) - f \left(x_p, y_q, t_{\frac{1}{2}}, \frac{3}{2} u_{p,q}^0 \right) \right] \right| + \left| T_{p,q}^{\frac{1}{2}} \right|$$

Since satisfies Lipchitz condition (3), we obtain

$$\left| e_{i,j}^1 \right| \leq \delta L \left| u(x_p, y_q, t_n) - u_{p,q}^0 \right| + \left| T_{p,q}^{\frac{1}{2}} \right|$$
$$= \delta L \left| e_{p,q}^0 \right| + \left| T_{p,q}^{\frac{1}{2}} \right| = \left| T_{p,q}^{\frac{1}{2}} \right|$$
$$\leq C(k^{2-\alpha} + h^2)$$

Thus $\left| e_{i,j}^1 \right| \leq \frac{2C}{(2^{1-\alpha} w_0)} (k^{2-\alpha} + h^2)$.

$$\|E^1\| \leq \left(\frac{C_1}{2^{1-\alpha} w_0} \right) (k^{2-\alpha} + h^2), \qquad C_1 = 2C$$

Now suppose that

$$\|E^s\| \leq \left(\frac{C_1}{2^{1-\alpha} w_{s-1}} \right) (k^{2-\alpha} + h^2), C_s > 0, \text{ for } s = 0,1,2,\ldots,n$$

Set $M_s = \frac{C_s}{2^{1-\alpha} w_{s-1}}$, let $M = Max \{C, C_1, C_2, \ldots, C_n\}$.

For n+1, Let $\|E^{n+1}\|_{\infty} = Max_{\substack{1 \leq i \leq I-1 \\ 1 \leq j \leq I-1}} \left| e_{i,j}^{n+1} \right| = \left| e_{p,q}^{n+1} \right|$

$$\left| e_{i,j}^{n+1} \right| \leq \left| e_{p,q}^{n+1} \right| = (1 + 2r) \left| e_{p,q}^{n+1} \right| - \frac{r}{2} \left[4 \left| e_{p,q}^{n+1} \right| \right].$$
$$\leq (1 + 2r) \left| e_{p,q}^{n+1} \right| - \frac{r}{2} \left[\left| e_{p+1,q}^{n+1} \right| + \left| e_{p-1,q}^{n+1} \right| + \left| e_{p,q+1}^{n+1} \right| + \left| e_{p,q-1}^{n+1} \right| \right] \tag{18}$$
$$\leq \left| (1 + 2r) e_{p,q}^{n+1} - \frac{r}{2} \left(e_{p+1,q}^{n+1} + e_{p-1,q}^{n+1} + e_{p,q+1}^{n+1} + e_{p,q-1}^{n+1} \right) \right|$$

From (16) and (18), it follows that

$$\left| e_{i,j}^{n+1} \right| \leq \left| (1 - 2^{1-\alpha} w_1 - 2r) e_{p,q}^n + \frac{r}{2} \left(e_{p+1,q}^n + e_{p-1,q}^n + e_{p,q+1}^n + e_{p,q-1}^n \right) \right| + 2^{1-\alpha} \sum_{s=1}^{n-1} (w_{n-s} - w_{n-s+1}) \left| e_{i,j}^s \right| + \delta \left| f \left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2} u(x_p, y_q, t_n) - \frac{1}{2} u(x_p, y_q, t_{n-1}) \right) - f \left(x_p, y_q, t_{n+\frac{1}{2}}, \frac{3}{2} u_{p,q}^n - \frac{1}{2} u_{p,q}^{n-1} \right) \right| + \left| T_{p,q}^{n+\frac{1}{2}} \right|.$$

Since satisfies Lipchitz condition (3), we obtain

$$\left| e_{i,j}^{n+1} \right| \leq (1 - 2^{1-\alpha} w_1 - 2r) \|E^n\|_{\infty} + 2r \|E^n\|_{\infty} + 2^{1-\alpha} \sum_{s=1}^{n-1} (w_{n-s} - w_{n-s+1}) \|E^s\|_{\infty} + \delta L \left[\frac{3}{2} \left| e_{p,q}^n \right| + \frac{1}{2} \left| e_{p,q}^{n-1} \right| \right] + \left| T_{p,q}^{n+\frac{1}{2}} \right|.$$
$$\leq (1 - 2^{1-\alpha} w_1) M_n (k^{2-\alpha} + h^2) + 2^{1-\alpha} \sum_{s=1}^{n-1} (w_{n-s} - w_{n-s+1}) M_s (k^{2-\alpha} + h^2) + \delta L \left[\frac{3}{2} M_n (k^{2-\alpha} + h^2)_{\infty} + \frac{1}{2} M_{n-1} (k^{2-\alpha} + h^2) \right] + C(k^{2-\alpha} + h^2).$$
$$\leq [1 - 2^{1-\alpha} w_1 + 2^{1-\alpha} \sum_{s=1}^{n-1} (w_{n-s} - w_{n-s+1}) + 2 \delta L + 2^{1-\alpha} w_n] \left(\frac{M}{2^{1-\alpha} w_n} \right) (k^{2-\alpha} + h^2).$$
$$= [1 + 2\delta L] \left(\frac{1}{2^{1-\alpha} w_n} \right) M(k^{2-\alpha} + h^2).$$

Thus $\|E^{n+1}\| \leq C_{n+1} \left(\frac{1}{2^{1-\alpha} w_n} \right) (k^{2-\alpha} + h^2)$, where $C_{n+1} = [1 + 2\delta L]M$.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, two numerical experiments are presented to show the efficacy and accuracy of the proposed method, with using Matlab (R2020a) software. For each example, we take different size-meshes: (I=5, 10, 20, 40). In order to make sure that the stability condition of Crank-Nicolson is satisfied and to increase the order of convergence, the time steps are chosen according to the following formula:

$$k \leq \left[\frac{h^2 (1 - 2^{(1-\alpha)} w_1)}{2^{(2-\alpha)} \Gamma(2-\alpha)} \right]^{\frac{1}{\alpha}} \tag{19}$$

where, $w_1 = \left[\left(\frac{3}{2} \right)^{(1-\alpha)} - \left(\frac{1}{2} \right)^{(1-\alpha)} \right]$

Clearly, with this time-stepping technique, the order of convergence for CN method is $O(h^2)$. However, with this formula, we cannot take a small value to α, because in this case

the time step becomes too small and that leads to a very large mesh-size with respect to time.

Moreover, we present the maximum absolute errors arise from using the proposed numerical schemes, using the formula: $E_{h,k} = \max\limits_{\substack{1 \le i \le I-1 \\ 1 \le n \le N-1}} |u(x_i, t_n) - u_i^n|$. In addition, the numerical order of convergence (NOC) is computed using the formula [16]:

$$S_{h,k} = \frac{\log\left(\frac{E_{2h,k}}{E_{h,k}}\right)}{\log(2)}$$

4.1 Numerical experiments

Example 4.1
Consider the following two-dimensional time fractional Semilinear diffusion equation:

$$u_t^\alpha = u_{xx} + u_{yy} + \left(\Gamma(2+\alpha)t - 2t^{(1+\alpha)}\right)e^{x+y} - e^{2(x+y)}t^{2(1+\alpha)} + u^2$$
$$0 < x < 1, 0 < y < 1, 0 < t < 1$$

with the initial-boundary conditions:

$$u(x,0,t) = e^x t^{1+\alpha}, u(x,1,t) = e^{1+x}t^{1+\alpha}, 0 < x < 1$$
$$u(0,y,t) = e^y t^{1+\alpha}, u(1,y,t) = e^{1+y}t^{1+\alpha}, 0 < y < 1,$$
$$u(x,y,0) = 0, 0 < x < 1, 0 < y < 1.$$

The exact solution is: $u(x,y,t) = e^{x+y}t^{1+\alpha}$.

In Table 1, we present the maximum absolute errors (MAE) and numerical order of convergence (NOC), and the central processing unit times (CPUTs) in second, arise from using linearly Crank-Nicolson method for example 4.1, for α=0.95 with taking different space and time steps. Moreover, to support the numerical findings, Figure 1 shows the numerical simulations of the exact, CNM solutions, for α=0.95, h=1/32.

Table 1. Maximum absolute errors and numerical order of convergence obtained from using linearly C.N. methods for example 4.1, with α=0.95

h	k	$E_{h,k}$	$S_{h,k}$	CPUT
1/4	0.0243	0.0052	……..	0.152891
1/8	0.0056	0.0013	2.0000	0.152856
1/16	0.0013	3.0186e-04	2.1066	1.815655
1/32	3.0497e-04	6.9768e-05	2.1132	219.918794
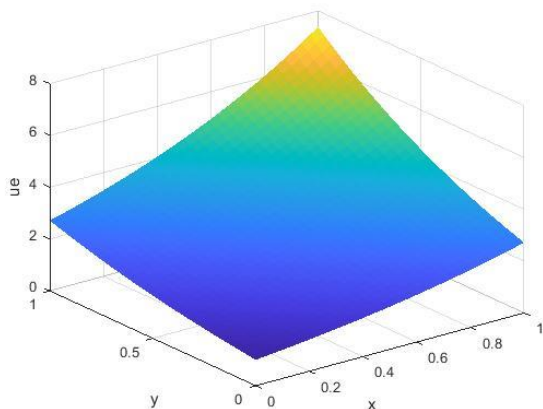
(a) Exact solution
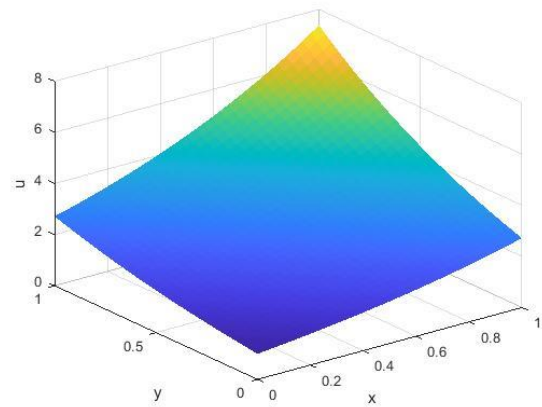
(b) CNM solution

Figure 1. Exact and CNM solutions for Example 4.1, with α=0.95, h=1/32, at t=1

Example 4.2
Consider the following two-dimensional time fractional Semilinear diffusion equation:

$$u_t^\alpha = u_{xx} + u_{yy} + \left(\frac{2t^{(2-\alpha)}}{\Gamma(3-\alpha)} - 2t^2\right)e^{x+y}$$
$$- t^3 e^{1.5(x+y)} + u^{\frac{3}{2}},$$
$$x, y \in (0,1), \ t \in (0,1),$$

with the initial-boundary conditions:

$$u(x,0,t) = e^x t^2, u(x,1,t) = e^{1+x}t^2,$$
$$x \in (0,1), t \in (0,1)$$
$$u(0,y,t) = e^y t^2, u(1,y,t) = e^{1+y}t^2,$$
$$y \in (0,1), t \in (0,1)$$
$$u(x,y,0) = 0, x, y \in (0,1),$$

The exact solution is: $u(x,y,t) = e^{x+y}t^2$

In Table 2, we present the maximum absolute errors (MAE) and numerical order of convergence (NOC), and the central processing unit times (CPUTs) in second, arise from using linearly Crank-Nicolson method for example 4.2, for α=0.9, with taking different space and time steps. Moreover, to support the numerical findings, Figure 2 shows the numerical simulations of the exact, CNM solutions, for α=0.9, h=1/32.
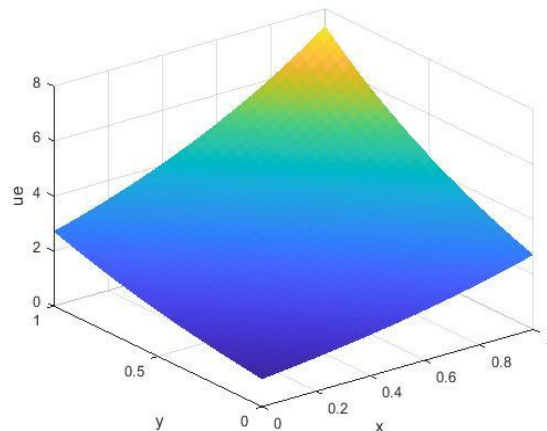
(a) Exact solution
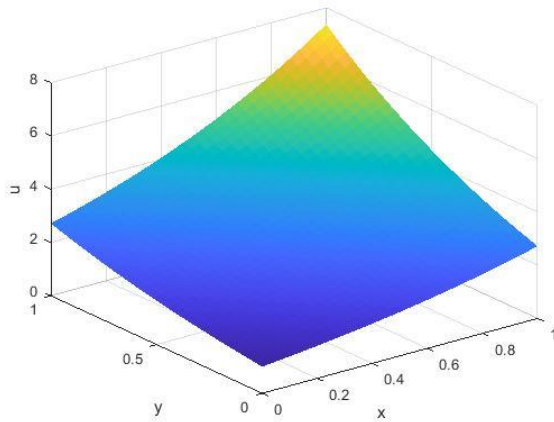
(b) CNM solution

Figure 2. Exact and CNM solutions for Example 4.2, with α=0.9, h=1/32, at t=1

Table 2. Maximum absolute errors and numerical order of convergence obtained from using linearly C.N. methods for example 4.2, with α=0.9

h	k	$E_{h,k}$	$S_{h,k}$	CPUT
1/4	0.0181	0.0038	……	0.108368
1/8	0.0039	8.9602e-04	2.0844	0.221917
1/16	8.3321e-04	2.0308e-04	2.1415	3.395496
1/32	1.7857e-04	4.5888e-05	2.1459	507.996491

4.2 Discussions of numerical results

Tables 1 and 2 show that the maximum absolute errors decrease, as we refine the space (time)-steps, which indicates that the numerical solution tends to the exact one. Hence the numerical convergence is satisfied, and that supports the theoretical results. In addition, with time-stepping formula (19), the numerical orders of convergence of CNM are larger than 2. Therefore, they are in a good agreement with the theoretical one. Furthermore, the required CPUTs increase, as we refine the space (time)-steps, and that because, when the space (time)-step is small, the number of required computational operations becomes large compared with taking large values to the space (time) steps. Moreover, from Figures 1 and 2, it can be easily noticed that the numerical solutions obtained from using CNM are in good agreement with and very close to the exact ones.

For most cases, the exact solution formula for the diffusion-reaction problem (2) cannot be found. Therefore, the proposed numerical technique can help us to indicate whether, the solution of the governed problem (2) stays bounded in the time interval (0, T), as in examples 4.1 and 4.2. If so, this physically means, throughout this period of time, in the spatial domain, the diffusion impact is greater than the effect of the reaction.

5. CONCLUSIONS

In this paper, the Crank-Nicolson numerical finite difference scheme is proposed to solve a two-dimensional time-fractional Semilinear parabolic equation with homogeneous Dirichlet boundary conditions. We prove that the proposed scheme is consistent, conditionally stable and convergent. In addition, in order to support the theoretical

findings, two numerical experiments are considered.

We observed that the order of convergence of Crank-Nicolson scheme, which takes the form: $O(k^{2-\alpha} + h^2)$, is even higher than the order of Euler implicit (explicit) scheme, which takes the form: $O(k + h^2)$ [18]. Moreover, the numerical results confirm that the proposed method is convergent and provides an accurate strategic solution with a high order of numerical convergence (greater than 2), compared with other finite difference schemes, such as Euler implicit (explicit) scheme, which their numerical order of convergence should be less than 2.

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