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On the Third Hankel Determinant of Certain Subclass of Bi-Univalent Functions

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ABSTRACT

In this study, we introduce a novel subclass of bi-univalent functions, which are of considerable interest in various fields of mathematics, including complex analysis and geometric function theory. By employing the property of subordination, we define these bi-univalent functions as $\mathcal{R}(t,\gamma,\lambda)$ and impose constraints on the coefficients $|a_n|$. Our investigation provides the upper bounds for the bi-univalent functions in this newly developed subclass, specifically for n=2, 3, 4, and 5. We then derive the third Hankel determinant for this particular class, which reveals several intriguing scenarios. These findings contribute to the broader understanding of bi-univalent functions and their potential applications in diverse mathematical contexts. Notably, the results obtained may serve as a foundation for future investigations into the properties and applications of bi-univalent functions and their subclasses.

1. INTRODUCTION

Let N refers to the collection of functions f analytic in the open unit disk $U=\{z: z\,\epsilon\,\mathbb{C}\ and\ |z|<1\}$. An analytic function $f\in N$ has Taylor series expansion of the form:

$$f(z)=z+\sum_{j=2}^{\infty}a_j z^j,\quad (\ z\in U). \tag{1}$$

The class of all functions in N refers by N_u which are univalent in U. The Koebe One-Quarter Theorem [1] ensures that the image of U under each $f\in N_u$ has a disk of radius ¼. Obviously, each $f\in N_u$ contains an inverse function f^{-1} is satisfying $f^{-1}(f(z))=z$ and $f^{-1}(f(w))=w$, $((|w|<r_0(f), r_0(f)\geq {}^1/_4))$, where,

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3 - (5a_2^3-5a_2a_3+a_4)w^4+\cdots,\quad (w\in U). \tag{2}$$

For two f and Φ be analytic functions, f is told to be subordinate to Φ in U and written as $f(z)\prec\Phi(z)$, if there exists a Schwarz function w be analytic such that $f(z)=\Phi(w(z))$ with $w(0)=0$ and $|w(z)|\leq 1$, $(z\in U)$. A function $f\in\Sigma$ is said to be bi- univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

In 1967, Lewin [2], for all function $f\in\Gamma$ of the form (1), and let Γ refer to the class of bi-univalent functions in U. The second coefficient of f satisfyies the estimate $|a_2|$<1.51. In 1967, Brannan and Clunie [3] conjectured that $|a_2|\leq\sqrt{2}$ for $f\in\Gamma$. After that Al-Shaqs [4] proved that $|a_2|=4/3$. In 1985 it was stated that Brannan-Clunie conject for bi-starlike function [5]. Brannan and Taha [6] gained evaluation estimates on the initial coefficients $|a_2|$ as well as $|a_3|$ for functions in the classes of bi- starlike functions of order ρ denoted $E_\Omega^*(\rho)$ and bi-convex functions of order ρ symbol $Y_\Omega(\rho)$. For all of the function classes $E_\Omega^*(\rho)$ and $Y_\Omega(\rho)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients were found in studies [6-10]. Many researchers [10-15] have studied numerous curious subclasses of the bi-univalent function class Ω and observed non-sharp bounds on the first two Taylor-Maclaurin coefficient. As well as the coefficient problem for all of the Taylor-Maclaurin coefficient $|a_n|$, $n=3,4,\ldots$ is as yet an open problem [2, 16].

Noonan and Thomas [3] defined q^{th} Hankel determinant of f, in 1976 for $n\geq 1$ and $q\geq 1$ by:

$$\mathrm{H}_q(n)=\begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2}\cdots & a_{n+q}\\ \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q}\cdots & a_{n+2q-2}\end{vmatrix},(a_1=1).$$

For $q=2$ and $n=1$, we know that the function $\mathrm{H}_2(1)=a_3-a_2^2$ .The second Hankel determinant $\mathrm{H}_2(2)$ is given through $|\mathrm{H}_2(2)|=|a_2a_4-a_3^2|$ for the classes of bi- starlike and bi-convex [17, 18] and third Hankel determinant, this functions are studied by Babalola [19] functional given by:

$$\mathrm{H}_3(1)=\begin{vmatrix} a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}\qquad (a_1=1)\ and\ (n=1, q=3).$$

By applying triangle inequality for $\mathrm{H}_3(1)$ we have:

$$|\mathrm{H}_3(1)|\leq|a_3||a_2a_4-a_3{}^2|-|a_4||a_4-a_2a_3| +|a_5||a_3-a_2{}^2|. \tag{3}$$

The Chebyshev polynomials are divided into four kinds. Chebyshev polynomials, which we include in this idea, play an important role in numerical analysis and mathematical physics. The majority of research papers on real orthogonal polynomials of the Chebyshev family include primarily findings of Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, as well as their numerous uses in diverse applications [20, 21]. Known widely sundry of the Chebyshev polynomials are the first and second kinds. In the case of real variable *x* on (1), the first and second kinds of Chebyshev polynomials that you are familiar with are: $T_n(x)=\cos(n \arccos x)$,

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)} = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}.$$

The function stated below was taken into consider ion in this work: $N(t,z) = \frac{1}{1-2tz+z^2}, t \in \left(\frac{1}{2},1\right), z \in U$.

It is familiar that if $t=\cos\rho$, $\rho \in \left(0,\frac{\pi}{3}\right)$. Then $N(t,z) = 1 + \sum_{n=1}^{\infty} \frac{\sin[(n+1)\rho]}{\sin\rho} z^n = 1 + 2\cos\rho z + (3\cos^2\rho - \sin^2\rho)z^2 + (8\cos^3\rho - 4\cos\rho)z^3 + \cdots$, $z \in U$, that is:

$$N(t,z) = 1 + U_1(t)\,z + U_2(t)z^2 + U_3(t)z^3 + U_4(t)z^4 + \cdots, t \in \left(\frac{1}{2},1\right), z \in U, \tag{4}$$

where,

$$U_n(t) = \frac{\sin[(n+1)\arccos t]}{\sqrt{1-t^2}}, n \in \mathbb{N},$$

where, $U_n(t)$ are the second kind Chebyshev polynomials. It is understood from the concept of the second kind Chebyshev polynomials that $U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t)$.

We get that

$$U_1(t) = 2t,\; U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t, \text{ (for each } n \in \mathbb{N}). \tag{5}$$

Lemma 1:[1] If P be a class of all analytic functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{6}$$

with p(0)=1 and $\Re e\{p(z)\} > 0$ for all z∈U. Then $|p_n| \leq 2$, for every (*n*=1,2,3,…). This disparity is sharp for each *n*.

Lemma 2: [22] If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some *x*, *z*, with $|x| \leq 1, |z| \leq 1$.

The aim of this idea is to evaluate the third Hankel determinant for class of bi-univalent functions.

2. MAIN RESULTS

Definition 1: A function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{R}(t,\gamma,\lambda)$, if the following conditions are satisfied holds:

$$\frac{1}{\gamma}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + zf''(z)\right\} \prec N(t,z), \tag{7}$$

and

$$\frac{1}{\gamma}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + wg''(w)\right\} \prec N(t,w), \tag{8}$$

where, $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda \geq 1, z, w \in U$ and $g = f^{-1}$, where the functional g is given by (2).

Remark 1: A function $f \in \Sigma$, given by (1) is said to be in the class $\mathcal{R}(t,\gamma,\lambda)$ $(\lambda = 1)$, if the following conditions are satisfied holds:

$$\frac{1}{\gamma}\{f'(z) + zf''(z)\} \prec N(t,z),$$

and

$$\frac{1}{\gamma}\{g'(w) + wg''(w)\} \prec N(t,w).$$

Remark 2: A function $f \in \Sigma$,, given by (1) is said to be in the class $\mathcal{R}(t,\gamma,\lambda), (\gamma = 1)$, if the following conditions are satisfied holds:

$$\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + zf''(z)\right\} \prec N(t,z),$$

and

$$\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + wg''(w\right\} \prec N(t,w).$$

Theorem 1: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,\lambda)$, $t \in \left(\frac{1}{2},1\right)$, then

$$|a_2a_4 - {a_3}^2| \leq \begin{cases} D(t,2-), & \text{if } \Lambda(\xi,t) \geq 0 \text{ and } \mathcal{C}(\xi,t) \geq 0, \\ \max\left\{\frac{4\gamma^2t^2}{(7+2\lambda)^2}, D(t,2-)\right\}, & \text{if } \Lambda(\xi,t) > 0 \text{ and } \mathcal{C}(\xi,t) < 0, \\ \frac{4\gamma^2t^2}{(7+2\lambda)^2} & \text{if } \Lambda(\xi,t) \leq 0 \text{ and } \mathcal{C}(\xi,t) \leq 0, \\ \max\{D(t,\tau_0), D(t,2-)\}, & \text{if } \Lambda(\xi,t) < 0 \text{ and } \mathcal{C}(\xi,t) > 0, \end{cases} \tag{9}$$

where

$$D(t,2-) = \frac{4\gamma^2t^2}{(7+2\lambda)^2} + \frac{\Lambda(\xi,t)+\mathcal{C}(\xi,t)}{2(3+\lambda)^4(7+2\gamma)^2(13+3\lambda)},$$

$$D(t,\tau_0) = \frac{4\gamma^2t^2}{(7+2\lambda)^2} - \frac{\mathcal{C}^2(\xi,t)}{8\Lambda(\xi,t)(3+\lambda)^4(7+2\lambda)^2(13+3\lambda)},$$

$$\tau_0 = \sqrt{\frac{-2\mathcal{C}(\xi,t)}{\Lambda(\xi,t)}},$$

$$\Lambda(\xi,t) = 2\gamma^2 t\{4(3+\lambda)^3(7+2\lambda)^2[4t^3 + 4t^2 - 3t - 1] + 2t(13+3\lambda)[2(3+\lambda)^4 - 10t(3+\lambda)^2(7+2\lambda)^2 - 8\gamma^2t^2(7+2\lambda)^2]\},$$

$$\mathcal{C}(\xi,t)=2\gamma^2\mathrm{t}(3+\lambda)^2\{4(3+\lambda)(7+2\lambda)^2[\mathrm{t}+4t^2-1]+2\mathrm{t}(13+3\lambda)[10t-4(3+\lambda)^2]\}.$$

Proof: Since $f\in\mathcal{R}(t,\gamma,\lambda)$. Then, we have

$$\frac{1}{\gamma}\left\{(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+zf''(z)\right\}=N\big(t,m(z)\big), \quad (10)$$

and

$$\frac{1}{\gamma}\left\{(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+wg''(w\right\}=N\big(t,n(w)\big). \quad (11)$$

Let $p,\ q\ \in P$ is defined as by follows

$$p(z)=\frac{1+\mathcal{S}(z)}{1-\mathcal{S}(z)}=1+\ p_1z+\ p_2z^2+\ p_3z^3+\cdots,$$

and

$$q(w)=\frac{1+\vartheta(w)}{1-\vartheta(w)}=1+q_1w+q_2w^2+q_3w^3+\cdots.$$

It follows that:

$$\mathcal{S}(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots, \quad (12)$$

and

$$\vartheta(w)=\left(\frac{q(w)-1}{q(w)+1}\right)=\frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)w^3+\cdots. \quad (13)$$

From (12) and (13), we take into account (5), we can easy to conclude

$$N\big(t,\mathcal{S}(z)\big)=1+\frac{U_1(t)}{2}p_1z+\left[\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2\right]z^2+\left[\frac{U_1(t)}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2(t)}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_3(t)}{8}p_1^3\right]z^3+\left[\frac{U_1(t)}{2}\left(p_4-\frac{p_2^2}{2}+p_1\left(\frac{p_1^3}{4}+\frac{3}{4}p_1p_2-p_3\right)\right)+\frac{U_2(t)}{2}\left(p_2^2+p_1\left(\frac{3\,p_1^3}{8}+\frac{3}{2}p_1p_2+\frac{p_1}{2}+\frac{p_3}{2}\right)-\frac{3}{8}U_3(t)p_1(p_1^3+p_1p_2)+U_4(t)p_1^4\right]z^4+\cdots, \quad (14)$$

and

$$N\big(t,\vartheta(w)\big)=1+\frac{U_1(t)}{2}q_1w+\left[\frac{U_1(t)}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_2(t)}{4}q_1^2\right]w^2+\left[\frac{U_1(t)}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{U_2(t)}{2}q_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_3(t)}{8}q_1^3\right]w^3+\left[\frac{U_1(t)}{2}\left(q_4-\frac{q_2^2}{2}+q_1\left(\frac{q_1^3}{4}+\frac{3}{4}q_1q_2-q_3\right)\right)+\frac{U_2(t)}{2}\left(q_2^2+q_1\left(\frac{3\,q_1^3}{8}+\frac{3}{2}q_1q_2+\frac{q_1}{2}+\frac{q_3}{2}\right)-\frac{3}{8}U_3(t)q_1(q_1^3+q_1q_2)+U_4(t)q_1^4\right]w^4+\cdots. \quad (15)$$

From (10) and (14), we obtain that

$$\frac{3+\lambda}{\gamma}a_2=\frac{U_1(t)}{2}p_1, \quad (16)$$

$$\frac{7+2\lambda}{\gamma}a_3=\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2 \quad (17)$$

$$\frac{13+3\lambda}{\gamma}a_4=\frac{U_1(t)}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{U_2(t)}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_3(t)}{8}p_1^3. \quad (18)$$

Also, from (11) and (15), we obtain that

$$-\frac{3+\lambda}{\gamma}a_2=\frac{U_1(t)}{2}q_1, \quad (19)$$

$$\frac{7+2\lambda}{\gamma}(2a_2^2-a_3)=\frac{U_1(t)}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_2(t)}{4}q_1^2, \quad (20)$$

$$-\frac{13+3\lambda}{\gamma}(5a_2^3-5a_2a_3+a_4)=\frac{U_1(t)}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{U_2(t)}{2}q_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_3(t)}{8}q_1^3. \quad (21)$$

From (16) and (19), we get

$$\mathrm{a}_2=\frac{\gamma\mathrm{U}_1(\mathrm{t})}{2(3+\lambda)}\mathrm{p}_1=\frac{-\gamma\mathrm{U}_1(\mathrm{t})}{2(3+\lambda)}\mathrm{q}_1, \quad (22)$$

It follows that its

$$p_{1=-}q_1. \quad (23)$$

Subtracting (20) and (17) and considering (22), we get

$$a_3=\frac{\gamma^2U_1^2(t)}{4(3+\lambda)^2}p_1^2+\frac{\gamma U_1(t)}{4(7+2\lambda)}(p_2-q_2). \quad (24)$$

Also, subtracting (21) from (18) and considering (22) and (24), we find that

$$\frac{1}{\gamma}[(16(13+3\lambda)a_4+8(13+3\lambda))(5a_2^3-5a_2a_3)]=4U_1(t)(p_3-q_3)+4\big(U_2(t)-U_1(t)\big)\mathrm{p}_1(\mathrm{p}_2+\mathrm{q}_2)+2\Big(\big(\mathrm{U}_1(\mathrm{t})-2\mathrm{U}_2(\mathrm{t})+\mathrm{U}_3(\mathrm{t})\big)\Big)\mathrm{p}_1^3,$$

then

$$a_4=\frac{-5}{2}(a_2^3-a_2a_3)+\frac{2\gamma U_1(t)(p_3-q_3)}{8(13+3\lambda)}+\frac{2\gamma\big(U_2(t)-U_1(t)\big)}{8(13+3\lambda)}\mathrm{p}_1(\mathrm{p}_2+\mathrm{q}_2)+\frac{\gamma\big(\mathrm{U}_1(\mathrm{t})-2\mathrm{U}_2(\mathrm{t})+\mathrm{U}_3(\mathrm{t})\big)}{8(13+3\lambda)}p_3,$$

since

$$\frac{-5}{2}\left(a_2^3-a_2a_3\right)=-\frac{5\gamma^3\mathrm{U_1}^3(\mathrm{t})}{16(3+\lambda)^3}\mathrm{p_1}^3+\frac{5\gamma^3\mathrm{U_1}^3(\mathrm{t})}{16(3+\lambda)^3}\mathrm{p_1}^3+\frac{5\gamma^2\mathrm{U}_1^2(\mathrm{t})\mathrm{p}_1\left(\mathrm{p}_2-\mathrm{q}_2\right)}{16(3+\lambda)(7+2\lambda)},$$

Thus

$$a_4=\frac{5\gamma^2\mathrm{U}_1^2(\mathrm{t})\mathrm{p}_1(\mathrm{p}_2-\mathrm{q}_2)}{16(3+\lambda)(7+2\lambda)}+\frac{\gamma\mathrm{U}_1(\mathrm{t})(\mathrm{p}_3-\mathrm{q}_3)}{4(13+3\lambda)}+\frac{\gamma(\mathrm{U}_2(\mathrm{t})-\mathrm{U}_1(\mathrm{t}))}{4(13+3\lambda)}\mathrm{p}_1(\mathrm{p}_2+\mathrm{q}_2)+\frac{\gamma\left(\mathrm{U}_1(\mathrm{t})-2\mathrm{U}_2(\mathrm{t})+\mathrm{U}_3(\mathrm{t})\right)}{8(13+3\lambda)}\mathrm{p}_1^3 \quad (25)$$

Thus, form (22), (24) and (25), we get

$$a_2a_4-{a_3}^2=\frac{5\gamma^3\mathrm{U}_1^3(\mathrm{t})\mathrm{p}_1^2(\mathrm{p}_2-\mathrm{q}_2)}{32(3+\lambda)^2(7+2\lambda)}+\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})\mathrm{p}_1(\mathrm{p}_3-\mathrm{q}_3)}{8(3+\lambda)(13+3\lambda)}+\frac{\gamma^2\mathrm{U}_1(\mathrm{t})(\mathrm{U}_2(\mathrm{t})-\mathrm{U}_1(\mathrm{t}))}{8(3+\lambda)(13+3\lambda)}\mathrm{p}_1^2(\mathrm{p}_2+\mathrm{q}_2)-\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})(\mathrm{p}_2-\mathrm{q}_2)^2}{16(7+2\lambda)^2}+\frac{\gamma^2\mathrm{U}_1(\mathrm{t})}{16(3+\lambda)^4(13+3\lambda)}\mathrm{p}_1^4\left[\left(\mathrm{U}_1(\mathrm{t})-2\mathrm{U}_2(\mathrm{t})+\mathrm{U}_3(\mathrm{t})\right)(3+\lambda)^3-\gamma^2\mathrm{U}_1^3(\mathrm{t})(13+3\lambda)\right]. \quad (26)$$

According to Lemma (2) we get

$$2p_2=p_1^2+(4-p_1^2)x \ \ and\ 2q_2=q_1^2+(4-q_1^2)y, \quad (27)$$

and

$$4\mathrm{p}_3=p_1^3+2(4-p_1^2)\mathrm{p}_1x-(4-p_1^2)\mathrm{p}_1x^2+2(4-p_1^2)(1-|x|^2)z, \quad (28)$$

and

$$4\mathrm{q}_3=q_1^3+2(4-q_1^2)\mathrm{q}_1y-(4-q_1^2)\mathrm{q}_1y^2+2(4-q_1^2)(1-|y|^2)w,$$

for x,y,z,w with $|x|\le 1, |y|\le 1, |z|\le 1, |w|\le 1$.

Since (23) $p_{1=-}q_1$, from (27) and (28) we get

$$\mathrm{p}_2-\mathrm{q}_2=\frac{4-p_1^2}{2}(x-y),\ \mathrm{p}_2+\mathrm{q}_{2=}p_1^2+\frac{4-p_1^2}{2}(x+y), \quad (29)$$

and

$$\mathrm{p}_3-\mathrm{q}_3=\frac{p_1^3}{2}+\frac{(4-p_1^2)\mathrm{p}_1}{2}(x+y)-\frac{(4-p_1^2)\mathrm{p}_1}{4}(x^2+y^2)+\frac{4-p_1^2}{2}[(1-|x|^2)z-(1-|y|^2)w]. \quad (30)$$

According to Lemma (2), we can draw assumptions without any restrictions that $\tau\in[0,2]$, where $\tau=|\mathrm{p}_1|$.

Thus, substituting the expressions (29) and (30) in (26) and utilize triangle inequality, letting $|\mathrm{x}|=\mathcal{J}$, $|\mathrm{y}|=\Upsilon$, we can easily obtain that

$$|a_2a_4-{a_3}^2|\le b_1(t,\tau)(\mathcal{J}+\Upsilon)^2+b_2(t,\tau)(\mathcal{J}^2+\Upsilon^2)+b_3(t,\tau)(\mathcal{J}+\Upsilon)+b_4(t,\tau):=F(\mathcal{J},\Upsilon), \quad (31)$$

where,

$$b_1(t,\tau)=\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})(4-t^2)^2}{64(7+2\lambda)^2}\ge 0,\ b_2(t,\tau)=\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})\,\mathrm{t}(\mathrm{t}-2)(4-t^2)}{32(3+\lambda)(13+3\lambda)}\le 0,$$

$$b_3(t,\tau)=\frac{5\gamma^3\mathrm{U}_1^3(\mathrm{t})t^2(4-t^2)}{64(3+\lambda)^2(7+2\lambda)}+\frac{\gamma^2\mathrm{U}_1(\mathrm{t})\mathrm{U}_2(\mathrm{t})t^2(4-t^2)}{16(3+\lambda)(13+3\lambda)}\ge 0,$$

$$b_4(t,\tau)=\frac{\gamma^2\mathrm{U}_1(\mathrm{t})[(3+\lambda)^3\mathrm{U}_3(\mathrm{t})-U_1^3(t)(13+3\lambda)]\,t^4}{16(3+\lambda)^4(13+3\lambda)}+\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})\,\mathrm{t}(4-t^2)}{8(3+\lambda)(13+3\lambda)}\ge 0,\ t\in\left(\frac{1}{2},1\right),\ \tau\in[0,2].$$

Therefore, we must maximize the function. $F(\mathcal{J}+\Upsilon)$ on the closed square.

$\Psi=\{(\mathcal{J}+\Upsilon):\mathcal{J},\Upsilon\in[0,1]\}$ for $\tau\in[0,2]$. Since the coefficients of the function $F(\mathcal{J},\Upsilon)$ is dependent to variable τ for fixed value of t, we must investigate the maximum of $F(\mathcal{J},\Upsilon)$ respect to τ taking account these cases τ=0, τ=2 and τ∈(0,2).

Let τ=0. Then, we write

$$F(\mathcal{J},\Upsilon)=b_1(t,0)=\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})}{4(7+2\lambda)^2}(\mathcal{J}+\Upsilon)^2.$$

It is obvious that the maximum of the function is $F(\mathcal{J},\Upsilon)$ occurs at $(\mathcal{J},\Upsilon)=(1,1)$, and

$$\max\{F(\mathcal{J},\Upsilon):\mathcal{J},\Upsilon\in[0,1]\}=F(1,1)=\frac{\gamma^2\mathrm{U}_1^2(\mathrm{t})}{(7+2\lambda)^2}. \quad (32)$$

Now, let τ=2. In this case, $F(\mathcal{J},\Upsilon)$ is constant function as follows:

$$F(\mathcal{J},\Upsilon)=b_4(t,2)=\frac{\gamma^2\mathrm{U}_1(\mathrm{t})[(3+\lambda)^3\mathrm{U}_3(\mathrm{t})-U_1^3(t)(13+3\lambda)]}{(3+\lambda)^4(13+3\lambda)} \quad (33)$$

In the case $\tau\in(0,2)$, we will examine the maximum of the function $F(\mathcal{J}+\eta)$ taking into account the sing of $\Gamma(\mathcal{J},\Upsilon)=F_{\mathcal{J}\mathcal{J}}(\mathcal{J},\Upsilon)F_{\Upsilon\Upsilon}(\mathcal{J},\Upsilon)-[F_{\mathcal{J}\Upsilon}(\mathcal{J},\Upsilon)]^2$. We can easily see that by simple computation, we can easily see that

$$\Gamma(\mathcal{J},\Upsilon)=4b_2(t,\tau)[2b_1(t,\tau)+b_2(t,\tau)].$$

Since $b_2(t,\tau)<0$ for all $t\in\left(\frac{1}{2},1\right)$, $\tau\in[0,2]$ and $2b_1(t,\tau)+b_2(t,\tau)>0$, we conclude that $\Gamma(\mathcal{J},\Upsilon)<0$. Therefore, the function F cannot have a local maximum in interior of the closed square Γ. Let

$$\partial\Gamma=\{(0,\Upsilon):\Upsilon\in[0,1]\}\cup\{(\mathcal{J},0):\mathcal{J}\in[0,1]\}\cup\{(1,\Upsilon):\Upsilon\in[0,1]\}\cup\{(\mathcal{J},1):\mathcal{J}\in[0,1]\}.$$

We may actually demonstrate that the function's maximum $F(\mathcal{J},\Upsilon)$ on the boundary ∂Γ of the square Γ value is occurs at $(\mathcal{J},\Upsilon)=(1,1)$, and

$$\max\{F(\mathcal{J},\Upsilon):(\mathcal{J},\Upsilon)\in\partial\Gamma\}=F(1,1)=4b_1(t,\tau)[2b_2(t,\tau)+b_3(t,\tau)]+b_4(t,\tau),\quad t\in\left(\frac{1}{2},1\right),\tau\in(0,2). \quad (34)$$

Let us now define the function $D:(0,2)\to\mathbb{R}$ as follows:

$$D(t,\tau)=4b_1(t,\tau)[2b_2(t,\tau)+b_3(t,\tau)]+b_4(t,\tau), \quad (35)$$

for fixed value of t.

Substituting the value $bj(t,\tau)$, j=1,2,3,4 in the (35), we get

$$D(t,\tau) = \frac{\Lambda(\xi,t)\tau^4+4\mathcal{C}(\xi,t)\tau^2}{32(3+\lambda)^4(7+2\lambda)^2(13+3\lambda)} + \frac{\delta^2 U_1^2(t)}{(7+2\lambda)^2},$$

where,

$$\Lambda(\xi,t) = \gamma^2 U_1(t)\{2(3+\lambda)^3(7+2\lambda)^2[-U_1(t)+2U_2(t)+U_3(t)] + U_1(t)(13+3\lambda)[2(3+\lambda)^4 - 5U_1(t)(3+\lambda)^2(7+2\lambda)^2 - 2\gamma^2 U_1^2(t)(7+2\lambda)^2]\},$$

$$\mathcal{C}(\xi,t) = \gamma^2 U_1(t)(3+\lambda)^2\{2(3+\lambda)(7+2\lambda)^2[U_1(t)+2U_2(t)] + U_1(t)(13+3\lambda)[5U_1(t)-4(3+\lambda)^2]\},$$

where,

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t),$$

We get that,

$$U_1(t) = 2t,\ U_2(t) = 4t^2-1,\ U_3(t) = 8t^3-4t. \text{ Then}$$
$$\Lambda(\xi,t) = 2\gamma^2 t\{4(3+\lambda)^3(7+2\lambda)^2[4t^3+4t^2-3t-1] + 2t(13+3\lambda)[2(3+\lambda)^4 - 10t(3+\lambda)^2(7+2\lambda)^2 - 8\gamma^2 t^2(7+2\lambda)^2]\},$$

$$\mathcal{C}(\xi,t) = 2\gamma^2 t(3+\lambda)^2\{4(3+\lambda)(7+2\lambda)^2[t+4t^2-1] + 2t(13+3\lambda)[10t-4(3+\lambda)^2]\}.$$

Now, suppose that H(t,τ) has maximum value in an interior of τ∈[0,2], then

$$D'(t,\tau) = \frac{\Lambda(\xi,t)\tau^2+\mathcal{C}(\xi,t)}{8(3+\lambda)^4(7+2\lambda)^2(13+3\lambda)}\tau.$$

After some calculations, we take into account the following cases:

Let $\Lambda(B,t) \geq 0$ and $\mathcal{C}(B,t) \geq 0$. Then $D'(t,\tau) \geq 0$,so $D(t,\tau)$ is an increasing function. Therefore,

$$\text{Max}\{D(t,\tau): \tau \in (0,2)\} = D(t,2-) = \frac{\Lambda(\xi,t)+\mathcal{C}(\xi,t)}{2(3+\lambda)^4(7+2\lambda)^2(13+3\lambda)} + \frac{\gamma^2 U_1^2(t)}{(7+2\lambda)^2}. \quad (36)$$

That is, max $\{\max\{F(\mathcal{J},\Upsilon): (\mathcal{J},\Upsilon) \in [0,1]\}: \tau \in (0,2)\} = H(t,2-)$.

1. Let $\Lambda(\xi,t) > 0$ and $\mathcal{C}(\xi,t) < 0$. Then $\tau_0 = \sqrt{\frac{-2\mathcal{C}(\xi,t)}{\Lambda(B,t)}}$ is a critical point of the function $D(t,\tau)$. We suppose that τ_0∈(0,2). Since $D''(t,\tau) > 0$, τ_0 is a local minimum point of the function $D(t,\tau)$. That is the function $D(t,\tau)$ cannot have a local maximum.
2. Let $\Lambda(\xi,t) \leq 0$ and $\mathcal{C}(\xi,t) \leq 0$. Then $D'(t,\tau) \leq 0$. Thus, $D(t,\tau)$ is a decreasing function on the interval (0,2). Therefore,

$$\max\{D(t,\tau): \tau \in (0,2)\} = D(t,0+) = 4b_1(t,0) = \frac{\gamma^2 U_1^2(t)}{(7+2\lambda)^2}. \quad (37)$$

1. Let $\Lambda(\xi,t) < 0$ and $\mathcal{C}(\xi,t) > 0$. Then τ_0 is a critical point of the function $D(t,\tau)$.
2. We suppose that τ_0∈(0,2). Since $D''(t,\tau) < 0$, τ_0 is a local maximum of the function $D(t,\tau)$ and maximum value occurs at $\tau=\tau_0$. Therefore,

$$\max\{D(t,\tau): \tau \in (0,2)\} = D(t,\tau_0), \quad (38)$$

where

$$D(t,\tau_0) = \frac{4\gamma^2 t^2}{(7+2\lambda)^2} - \frac{\mathcal{C}^2(\xi,t)}{8\Lambda(\xi,t)(3+\lambda)^4(7+2\lambda)^2(13+3\lambda)}.$$

Thus from (32) to (38), the proof of Theorem (1) is complete.

Theorem 2: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,\lambda)$, $t \in \left(\frac{1}{2},1\right)$, then

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{t^3\gamma^3(13+3\lambda) - \gamma(3+\lambda)^3(8t^3-8t^2-2t-4)}{(3+\lambda)^3(13+3\lambda)} & \omega \leq \tau \leq 2 \\ \frac{t\gamma}{2(13+3\lambda)} & 0 \leq \tau \leq \omega \end{cases}, \quad (39)$$

where

$$\omega = \frac{d_2 + \sqrt{{d_2}^2 + 12(d_3-d_1)}}{3(d_3-d_1)}.$$
$$d_1 = \frac{14\gamma^2}{16(3+\lambda)(7+2\lambda)} - \frac{\gamma(4t^2-2t-1)}{4(13+3\lambda)},\ d_2 = \frac{t\gamma}{8(13+3\lambda)}$$

and

$$d_3 = \frac{t^3\gamma^3(13+3\lambda) - \gamma(3+\lambda)^3(8t^3-8t^2-2t-4)}{(3+\lambda)^3(13+3\lambda)}.$$

Proof: From (22), (24) and (25), we get

$$|a_2a_3 - a_4| = \left| \frac{\gamma^3 {U_1}^3(t)(13+3\lambda) - \gamma(3+\lambda)^3(U_1(t)-2U_2(t)+U_3(t))}{8(3+\lambda)^3(13+3\lambda)} p_1^3 + \frac{7\gamma^2 U_1^2(t) p_1(p_2-q_2)}{8(3+\lambda)(7+2\lambda)} + \frac{\gamma U_1(t)(p_3-q_3)}{4(13+3\lambda)} - \frac{\gamma(U_2(t)-U_1(t))}{4(13+3\lambda)} p_1^2(p_2+q_2) \right|.$$

According to Lemma 2, we assume without any restriction that $\tau \in [0,2]$, where $\tau = |p_1|$ thus for $\eta_1 = |x| \leq 1, \eta_2 = |y| \leq 1$, we get

$$|a_2a_3 - a_4| \leq d_1(\eta_1+\eta_2) + d_2({\eta_1}^2+{\eta_2}^2) + d_3 = R(\eta_1,\eta_2),$$

where,

$$d_1 = \frac{7\gamma^2 U_1^2(t)\tau(4-\tau^2)}{32(3+\lambda)(7+2\lambda)} \geq 0,\ d_2 = \frac{\gamma U_1(t)(\tau-2)(4-\tau^2)}{16(13+3\lambda)} \geq 0,$$
$$d_3 = \frac{\gamma^3 {U_1}^3(t)(13+3\lambda) - \gamma(3+\lambda)^3(U_1(t)-2U_2(t)+U_3(t))}{8(3+\lambda)^3(13+3\lambda)}\tau^3 - \frac{\gamma(U_2(t)-U_1(t))\tau(4-\tau^2)}{4(13+3\lambda)} \geq 0,\ t\left(\frac{1}{2},1\right), \tau \in [0,2].$$

By using the same method of Theorem 1, thus maximize occur at η1=1 and η2=1 in closed square [0,2],

$$R(\tau) = 4d_1 + 2[d_2 + d_3].$$

Substituting the value of d_1, d_2, d_3 in $R(\tau)$, we get that

$$R(\tau) = d_1\tau(4-\tau^2) - d_2(4-\tau^2) + d_3\tau^3,$$

so that

$$R(\tau) = 4d_1\tau - d_1\tau^3 - 4d_2 + d_2\tau^2 + d_3\tau^3,$$

where

$$d_1 = \frac{7\gamma^2 U_1^2(t)}{32(3+\lambda)(7+2\lambda)} - \frac{\gamma\big(U_2(t) - U_1(t)\big)}{4(13+3\lambda)},$$
$$d_2 = \frac{\gamma U_1(t)}{16(13+3\lambda)}$$

$$d_3 = \frac{\gamma^3 U_1{}^3(t)(13+3\lambda) - \gamma(3+\lambda)^3\big(U_1(t) - 2U_2(t) + U_3(t)\big)}{8(3+\lambda)^3(13+3\lambda)}.$$

Since

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t),$$

We get that,

$$U_1(t) = 2t,\ U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t.$$

Then

$$d_1 = \frac{14\gamma^2}{16(3+\lambda)(7+2\lambda)} - \frac{\gamma(4t^2 - 2t - 1)}{4(13+3\lambda)},$$
$$d_2 = \frac{t\gamma}{8(13+3\lambda)}$$

and

$$d_3 = \frac{t^3\gamma^3(13+3\lambda) - \gamma(3+\lambda)^3(8t^3 - 8t^2 - 2t - 4)}{(3+\lambda)^3(13+3\lambda)}.$$

We obtain

$$R'(\tau) = 3(d_3 - d_1)\tau^2 + 2d_2\tau + 4d_1, \qquad R''(\tau) = 6(d_3 - d_1)\tau + 2d_2,$$

If $d_3 - d_1 > 0$, that is $d_3 > d_1$, then we have $R'(\tau) > 0$. Thus $R(\tau)$ is an increasing function on the closed interval [0,2] and so the function $R(\tau)$ get the maximum value at $\tau=2$, that is

$$|a_2a_3 - a_4| \le R(2) = \frac{t^3\gamma^3(13+3\lambda) - \gamma(3+\lambda)^3(8t^3 - 8t^2 - 2t - 4)}{(3+\lambda)^3(13+3\lambda)}.$$

If $d_3 - d_1 < 0$, let $R'(\tau) = 0$, then we have

$$\tau = \omega = \frac{d_2 + \sqrt{d_2{}^2 + 12(d_3 - d_1)}}{3(d_3 - d_1)},$$

when $\omega<\tau\le 2$. Then we get $R'(\tau) > 0$, which means the function on the closed interval [0,2], thus $R(\tau)$ gets the maximum value at $R(2)$, which means the function $R(\tau)$ is an increasing function the closed interval [0,2], thus $R(\tau)$ gets the maximum value at $\tau=0$, we have

$$|a_2a_3 - a_4| \le R(0) = \frac{t\gamma}{2(13+3\lambda)}.$$

The proof of Theorem 2 is complete.

Theorem 3: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,\lambda)$, $t \in \left(\frac{1}{2}, 1\right)$, then

$$|a_3 - a_2{}^2| \le \frac{2\gamma t}{(7+2\lambda)}, \tag{40}$$

$$|a_3| \le \frac{4t^2\gamma^2}{(3+\lambda)^2}. \tag{41}$$

Proof: By using (24) and Lemma 1, we get that

$$|a_3| \le \frac{\gamma^2 U_1^2(t)}{(3+\lambda)^2}, \tag{42}$$

since

$$U_1(t) = 2t, \tag{43}$$

substituting (43) in (42), we get (41).

The following Fekete-Szegö functional, for $\mu \in \mathbb{C}$, $f \in \mathcal{R}(t,\gamma,\lambda)$

$$a_3 - \mu a_2{}^2 = \frac{(1-\mu)\gamma^2 U_1^2(t) p_1^2}{4(3+\lambda)^2} + \frac{\gamma U_1(t)}{4(7+2\lambda)}(p_2 - q_2)$$
$$= \frac{\gamma U_1(t)}{4}\left[\frac{(1-\mu)\gamma U_1(t)}{(3+\lambda)^2} p_1^2 + \frac{1}{(7+2\lambda)} p_2 - \frac{1}{(7+2\lambda)} q_2\right],$$

since $U_1(t)=2t$, then

$$a_3 - \mu a_2{}^2 = \frac{\gamma t}{2}\left[\frac{1}{(7+2\lambda)} p_2 + s(\mu) p_1^2 - \frac{1}{(7+2\lambda)} q_2\right],$$

where

$$s(\mu) = \frac{(1-\mu)2\gamma t}{(3+\lambda)^2}.$$

We conclude that

$$|a_3 - \mu a_2{}^2| \le \begin{cases} \{2\gamma t\, |s(\mu)| & |s(\mu)| \ge \dfrac{1}{(7+2\lambda)}, \\ \dfrac{2\gamma t}{(7+2\lambda)} & 0 \le |s(\mu)| \le \dfrac{1}{(7+2\lambda)}, \end{cases}$$

for $\mu=1$, we get (40).

Theorem 4: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,\lambda)$, $t \in \left(\frac{1}{2}, 1\right)$, then

$$|a_4| \le \frac{\gamma(8t^3 - 4t^2 - 5t + 2)}{(13+3\lambda)}, \tag{44}$$

$$|a_5| \leq \frac{6\gamma^2[(3+\lambda)(4t^3-2t^2-t)+(8t^4-8t^3-2t^2-2t)]}{(3+\lambda)^2(13+3\lambda)} - \frac{4\gamma[3(3+\lambda)^4(8t^3-16t^2+4)+16\gamma t(21+4\lambda)]}{(3+\lambda)^4(21+4\lambda)}. \quad (45)$$

Proof: By (25), we have

$$a_4 = \frac{5\gamma^2 U_1^2(t)p_1(p_2-q_2)}{16(3+\lambda)(7+2\lambda)} + \frac{\gamma U_1(t)(p_3-q_3)}{4(13+3\lambda)} + \frac{\gamma(U_2(t)-U_1(t))}{4(13+3\lambda)}p_1(p_2+q_2) + \frac{\gamma\left(U_1(t)-2U_2(t)+U_3(t)\right)}{8(13+3\lambda)}p_1^3.$$

Since

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t),$$

we get that,

$U_1(t) = 2t,\ U_2(t) = 4t^2-1,\ U_3(t) = 8t^3-4t,$
and Lemma 1, we get (44).

From (10) and (14), we obtain that

$$\frac{(21+4\lambda)}{\gamma}a_5 = \frac{U_1(t)}{2}\left(p_4 - \frac{p_2^2}{2} + p_1\left(\frac{p_1^3}{4} + \frac{3}{4}p_1p_2 - p_3\right)\right) + \frac{U_2(t)}{2}\left(p_2^2 + p_1\left(\frac{3\,p_1^3}{8} + \frac{3}{2}p_1p_2 + \frac{p_1}{2} + \frac{p_3}{2}\right)\right) - \frac{3}{8}U_3(t)p_1(p_1^3+p_1p_2) + U_4(t)p_1^4. \quad (46)$$

also, from (11) and (15), we have

$$\frac{(21+4\lambda)}{\gamma}(14{a_2}^4 + 3{a_3}^2 - 21{a_2}^2a_3 + 6a_2a_4 - a_5) = \frac{U_1(t)}{2}\left(q_4 - \frac{q_2^2}{2} + q_1\left(\frac{q_1^3}{4} + \frac{3}{4}q_1q_2 - q_3\right)\right) + \frac{U_2(t)}{2}\left(q_2^2 + q_1\left(\frac{3\,q_1^3}{8} + \frac{3}{2}q_1q_2 + \frac{q_1}{2} + \frac{q_3}{2}\right)\right) - \frac{3}{8}U_3(t)q_1(q_1^3+q_1q_2) + U_4(t)q_1^4. \quad (47)$$

Subtracting (47) from (46), we get that

$$a_5 = \frac{\gamma U_1(t)}{(21+4\lambda)}(p_4-q_4) + \frac{3\gamma(3+\lambda)(13+3\lambda)\left(2U_2(t)-U_1(t)\right) + 3\gamma^2U_1(t)(21+4\lambda)\left(U_2(t)-U_1(t)\right)p_1^2}{4(3+\lambda)(13+3\lambda)(21+4\lambda)}(p_2+q_2)$$
$$+ \frac{2\gamma(13+3\lambda)(21+4\lambda)\left(U_2(t)-2U_1(t)\right) + 3\gamma^2(21+4\lambda)U_1(t)p_1}{4(3+\lambda)(13+3\lambda)(21+4\lambda)}(p_3-q_3) - \frac{3\gamma U_3(t)p_1}{4(21+3\lambda)}(p_2+q_2)$$
$$- \frac{63\gamma^3{U_1}^3(t)p_1^2}{16(3+\lambda)^2(7+2\lambda)}(p_2-q_2) + \frac{3\gamma^2{U_1}^2(t)}{16(7+2\lambda)}(p_2-q_2)^2$$
$$+ \frac{3\gamma^2(3+\lambda)^3U_1(t)\left(U_1(t)-2U_2(t)+U_3(t)\right) - 4\gamma^4(13+3\lambda){U_1}^4(t)}{16(3+\lambda)^2(7+2\lambda)}p_1^4.$$

Since

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t),$$

we get that,

$U_1(t) = 2t$, $U_2(t) = 4t^2-1$, $U_3(t) = 8t^3-4t$, and Lemma 1, we get (45).

Theorem 5: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,\lambda)$, $t \in \left(\frac{1}{2},1\right)$, $\gamma \in \mathbb{C}\setminus\{0\}$ and $\lambda \geq 1$, then we have

$$H_3(1) \leq \begin{cases} N\left(D(t,2-)\right) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) \geq 0 \text{ and } \mathcal{C}(\xi,t) \geq 0, \\ N\left(\max\left\{\frac{4\gamma^2t^2}{(7+2\lambda)^2}, D(t,2-)\right\}\right) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) > 0 \text{ and } \mathcal{C}(\xi,t) < 0, \\ N\left(\frac{4\gamma^2t^2}{(7+2\lambda)^2}\right) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) \leq 0 \text{ and } \mathcal{C}(\xi,t) \leq 0, \\ N(\max\{D(t,\tau_0), D(t,2-)\}) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) < 0 \text{ and } \mathcal{C}(\xi,t) > 0, \end{cases} \quad (48)$$

where, N, N_1, N_2, N_3, N_4 *and* $(\Lambda(\xi,t), \mathcal{C}(\xi,t))$ are given by (41), (44), (39), (45), (40) and (9) respectively.

Proof: Since

$H_3(1) = a_3(a_2a_4 - {a_3}^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - {a_2}^2)$, by applying the triangle inequality, we get

$$|H_3(1)| \leq |a_3||a_2a_4 - {a_3}^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - {a_2}^2|. \quad (49)$$

Substituting (41), (44), (39), (45), (40) and (9) in (49), we get (48).

Corollary 1: If f is given by (1) belongs to the subclass $\mathcal{R}(t,\gamma,1)$, $t \in \left(\frac{1}{2},1\right)$ and $\gamma \in \mathbb{C}\setminus\{0\}$, then we have

$$H_3(1) \leq \begin{cases} N(D(t,2-)) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) \geq 0 \text{ and } \mathcal{C}(\xi,t) \geq 0, \\ N(\max\left\{\frac{4\gamma^2t^2}{(7+2\lambda)^2}, D(t,2-)\right\}) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) > 0 \text{ and } \mathcal{C}(\xi,t) < 0, \\ N(\frac{4\gamma^2t^2}{(7+2\lambda)^2}) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) \leq 0 \text{ and } \mathcal{C}(\xi,t) \leq 0, \\ N(\max\{D(t,\tau_0), D(t,2-)\}) - N_1N_2 - N_3N_4, & \text{if } \Lambda(\xi,t) < 0 \text{ and } \mathcal{C}(\xi,t) > 0, \end{cases} \quad (50)$$

where, N, N_1, N_2, N_3, N_4 and $(\Lambda(\xi,t), \mathcal{C}(\xi,t))$ are given by (41), (44), (39), (45), (40) and (9) respectively with (*λ=1*).

3. CONCLUSIONS

This article presented a comprehensive investigation of the third Hankel determinant H3(1) for a novel subclass of bi-univalent functions, $\mathcal{R}(t,\gamma,\lambda)$. This subclass is of significant interest in various mathematical fields, including complex analysis and geometric function theory. Utilizing the property of subordination, we defined the bi-univalent functions $\mathcal{R}(t,\gamma,\lambda)$ and imposed constraints on the coefficients $|a_n|$. Our findings provided the upper bounds for the bi-univalent functions in this newly developed subclass, specifically for *n*=2, 3, 4, and 5. Furthermore, we advanced the understanding of these functions by deriving the third Hankel determinant for this particular class, which revealed several intriguing scenarios. This achievement led to the improvement of the bound of the third Hankel determinant for the class of bi-univalent functions $\mathcal{R}(t,\gamma,\lambda)$. Our study contributes to the broader understanding of bi-univalent functions, their subclasses, and their potential applications in diverse mathematical contexts. The results obtained may serve as a foundation for future investigations into the properties and applications of bi-univalent functions and their subclasses. Future research endeavors could explore further refinements of the bounds, as well as examine other subclasses of bi-univalent functions to uncover novel insights into their characteristics and potential applications. Ultimately, this study paves the way for a deeper exploration of the fascinating world of bi-univalent functions and their role in the realm of mathematics.

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