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Pseudospectral Method for Free Vibration Analysis of Vertically Standing Plates

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ABSTRACT

Vibration analysis of vertically standing plates subject to gravitational forces is crucial for designing walls, panels, and windows in building structures. This paper investigates the fundamental frequency of a vertically oriented, heavy plate with simply-supported edges. A pseudospectral method, a robust numerical technique, is employed to solve the governing differential equation incorporating various boundary conditions. The study presents frequency parameter values for a range of weight and width parameters. Additionally, the numerical technique is extended to determine the frequencies of a plate on a Winkler foundation. Comparative assessments are conducted to validate the accuracy and reliability of the proposed method.

1. INTRODUCTION

Vertically-oriented plated structures are extensively employed in various engineering applications, including curtain facades, walls, panels, and windows of buildings, as well as in propelling missiles and rockets [1]. In these cases, the standing plate is subjected to its own weight, and accelerating mobile structures can generate body forces equivalent to gravity [2]. The body forces developed in the mid-plane of the plate, due to its weight or acceleration in its plane, influence the plate's stability and natural frequencies. Consequently, accounting for the effects of gravity on the vibration characteristics of these plates is crucial for designing such structures.

Free vibration analysis of rectangular plates represents a classical structural mechanics problem that holds particular interest for professionals in mechanical, civil, and aerospace engineering fields. The free vibration characteristics of thin, uniform-thickness rectangular plates have been comprehensively reported by Leissa [3]. Nonetheless, the introduction of complicating effects and additional geometric or material parameters renders the vibration characteristics data virtually limitless [4]. While vibrations of rectangular plates with varying properties are well-studied, the vibration analysis of standing vertical plates remains relatively scarce [1, 5].

Large standing plates with simply supported vertical sides are frequently utilized in walls and windows of buildings [5]. For heavy plates, the effect of gravity is a significant factor in their design. Moreover, accelerating mobile structures can generate body forces equivalent to gravity [6]. Therefore, investigating the vibration characteristics of heavy standing plates under their own weight can enhance the understanding and applicability of these structures in practical engineering designs [1].

Herrmann [7] is among the early references that employed an energy method to study specific cases related to the vibration of standing rectangular plates. Yu and Wang [5] used the semi-analytical Levy interpolation method to determine the fundamental frequency of standing plates with vertical simply supported edges. They applied the Levy separation method to reduce the governing equation to an ordinary differential equation with non-constant coefficients. The two-point boundary value problem was subsequently transformed into two deterministic initial value problems. This method was further employed in study [8] to determine the fundamental frequencies of a standing plate with simply supported edges and weakened by a horizontal internal hinge. Lai and Xiang [1] adopted the Discrete Singular Convolution (DSC) method to examine the influence of body forces on the buckling and vibration behavior of elastically restrained vertical plates. Recently, Guguloth et al. [9] presented a free vibration analysis of simply supported rectangular plates using ANSYS software, although weight was neglected.

This paper aims to apply the pseudospectral method for studying the vibration characteristics of a standing plate with simply supported vertical sides. Conventional spectral collocation methods that utilize differentiation matrices based on Lagrange interpolating polynomials encounter difficulties when imposing two boundary conditions at one node. Several approaches have been proposed by researchers to address this issue [10]. Numerous recursion codes and packages implementing spectral collocation methods exist [11-13]; however, incorporating general boundary conditions into the formulation remains challenging [14]. The proposed formulation overcomes the problem associated with the imposition of two boundary conditions at one node. The technique is first applied to study the free vibration of a heavy standing plate and subsequently employed to determine the frequencies of a plate on a Winkler foundation. The obtained results are compared with the semi-analytical method [5] and differential transform method [15]. The motivation for utilizing this approach lies in its numerical stability and flexible implementation for vibration analysis.

2. MATHEMATICAL MODELING

Consider an isotropic standing rectangular plate of height L and width aL and uniform thickness h, simply supported on the vertical sides with horizontal and vertical sides parallel to x,y axis respectively in the x-y plane as shown in Figure 1.

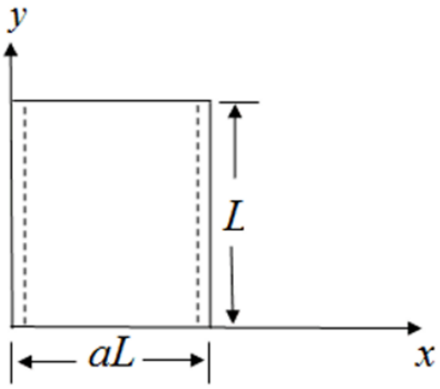

Figure 1. A rectangular plate with simply supported vertical edges

The bottom edge, bearing the total weight is clamped or simply supported and the top edge bearing no load is either free or simply supported. By simply supported, we mean a plate boundary that is prevented from deflecting but free to rotate about a line along the boundary edge. In other words, for a simply supported edge, the displacement is zero and the moment perpendicular to edge is also zero. A clamped edge in a plate is an edge wherein both the deflection and its slope are absent normally at an edge of a plate, a twisting moment, a bending moment and transverse shear force act. An edge for which all of these stress resultants vanish is considered to be a free edge. Adopting the classical plate theory and normalizing all lengths by the plate height L, the governing equation [16] is

$$\nabla^4 w+\gamma\frac{\partial}{\partial y}\left[(1-y)\frac{\partial w}{\partial y}\right]-K^4 w=0 \tag{1}$$

where, ∇^4 is the biharmonic differential operator. w is the lateral deflection, ρ the mass per unit area, g the gravitational acceleration, γ the weight parameter, K the frequency parameter.

$$\gamma=\frac{\rho g L^3}{D},\ K^4=\frac{\rho\Omega^2 L^4}{D},\ D=\frac{E h^3}{12\left(1-\upsilon^2\right)}$$

where, D is the flexural rigidity and Ω the frequency and υ is the Poisson's ratio. As the vertical sides are simply supported, Levy separation of variables is used wherein $w(x,y)=\sin\alpha x Y(y)$.

where $\alpha=\frac{n\pi}{a}, n$ an integer. Eq. (1) becomes

$$Y''''+Y''\left(-2\alpha^2+\gamma(1-y)\right)-\gamma Y'+\left(\alpha^4-K^4\right)Y=0 \tag{2}$$

where, the primes denote derivatives with respect to y. The boundary conditions to be satisfied at the horizontal edges are:

$$Y=\frac{dY}{dy}=0 \text{ for clamped edge} \tag{3}$$

$$\begin{aligned}&\frac{d^2Y}{dy^2}-\alpha^2 vY=0;\\ &\frac{d^3Y}{dy^3}-\alpha^2(2-v)\frac{dY}{dy}=0\end{aligned} \quad \text{for free edge} \tag{4}$$

$$Y=\frac{d^2Y}{dy^2}=0 \text{ for simply supported edge} \tag{5}$$

The boundary value problem is difficult to solve, even numerically [5]. Here a novel pseudospectral method is employed to numerically solve the problem. Eq. (2) is to be solved with the boundary conditions at $y=0$ and $y=1$. When the weight is absent and the plate is resting on a Winkler foundation with K_w being the foundation modulus parameter, Eq. (2) becomes

$$Y''''-2\alpha^2 Y''+\alpha^4 Y+K_w Y=K^4 Y \tag{6}$$

Plates on an elastic foundation are common structural elements that are widely employed in many civil engineering applications and Winkler model is supposed to be simplest model for an elastic foundation. The model assumed that the vertical displacement and pressure underneath it is linearly related to each other. The foundation reaction is included in the governing differential equation of the plate through the foundation parameter (K_w). The boundary value problems given by Eq. (2) and Eq. (6) are to be solved subject to the boundary conditions Eqns. (3)-(5) as required.

The methodology of solving the boundary value problems using the proposed pseudospectral method is outlined in the next section.

3. THE PSEUDOSPECTRAL METHOD

The pseudospectral method can be implemented using several approaches. In the literature, the approach of Fornberg [11] and the differentiation matrices approach [12, 13] are generally employed. An analysis of the numerical instabilities that may occur as the order of the derivative and the number of nodes are increased has been presented by Sadiq and Viswanath [17]. In addition, the difficulty in the incorporation of different boundary conditions in these approaches have led us to develop an approach of pseudospectral method that is simple to use and efficient in implementation. The methodology developed is particularly useful in solving vibration problems of rods, beams and plates of different geometries and configurations. The general steps in the proposed method can be outlined as follows: The physical domain $0\le y\le 1$ is first transformed to the computational domain $-1\le t\le 1$ using the transformation $t=2y-1$. With this transformation, $D_y^{(m)}=2^m D_t^{(m)}, m=1,2,3,4$ where $D_y^{(m)}$ is the differential operator with the subscript denoting the differentiation variable and super script in bracket denoting the order of differentiation. To apply the collocation technique, we assume

$$Y=\sum_{k=0}^{N} a_k T_k(t) \tag{7}$$

where, $T_k(t)$, (k=0,1,2, …) are Chebyshev polynomials that can be described as

$$T_k(t)=\cos\left(k\cos^{-1} t\right), \quad -1\le t\le 1$$

$$T_k(t)=\cos(k\theta) \text{ where } \theta = cos^{-1}\, t$$

The transformation $t=cos\theta$ is exploited here to evaluate the derivative directly in terms of the cosine and sine. This change of coordinates trick can reduce the error for a given number of grid points [18]. The grid points chosen in the paper are the Chebyshev-Gauss-Lobatto (CGL) nodes whose Lebesgue constant is very close to the optimal. In solving boundary value problems, it is opined [19] that CGL nodes often yield the best results. In addition, in the implementation of derivatives of Chebyshev polynomials using recurrence relations, the trigonometric derivative formulas are simpler to use and require fewer loops. This is used in the present work.

The derivatives are given by:

For $-1<t<1$;

$$\frac{d}{dt}T_k(t)=\frac{k\sin(k\theta)}{\sin\theta} \tag{8}$$

$$\frac{d^2}{dt^2}T_k(t)=\frac{-k^2\cos(k\theta)}{\sin^2\theta}+\frac{k\cos\theta}{\sin^3\theta}\sin k\theta \tag{9}$$

$$\frac{d^3T_k(t)}{dt^3}=\frac{1}{\sin^5\theta}\begin{pmatrix}\sin^2\theta \\ \left(k\sin(k\theta)-k^3\sin(k\theta)\right) \\ +3k\sin(k\theta)\cos^2\theta \\ -3k^2\cos(k\theta)\cos\theta\sin\theta\end{pmatrix} \tag{10}$$

$$\frac{d^4T_k(t)}{dt^4}=\frac{\begin{bmatrix}\sin^2\theta\begin{pmatrix}9k\sin(k\theta)\cos\theta \\ -6k^3\sin(k\theta)\cos\theta\end{pmatrix} \\ -\sin^3\theta\begin{pmatrix}4k^2\cos(k\theta)- \\ k^4\cos(k\theta)\end{pmatrix} \\ +15k\sin(k\theta)\cos^3\theta \\ -15k^2\cos(k\theta)\cos^2\theta\sin\theta\end{bmatrix}}{\sin^7\theta} \tag{11}$$

and

$$\frac{d^n}{dt^n}T_k(t)=(\pm1)^{k+n}\prod_{p=0}^{n-1}\frac{k^2-p^2}{2p+1}\text{, at t}=\pm1 \tag{12}$$

In the collocation framework, we select N-3 points in $(0, \pi)$ and require $Y(\theta)$ to satisfy Eq. (2) at these N-3 points in addition to satisfying the boundary conditions at the end points. The internal points which are the extrema of $T_N(t)$ are given by:

$$t_i=\cos\left(\frac{(N-i)\,\pi}{N}\right),\ i=1,2,\ldots,N-3 \tag{13}$$

that corresponds to the points

$$\theta_i=\frac{(N-i)\pi}{N}\ ,\ i=1,2,\ldots,N-3 \text{ in } (0,\pi) \tag{14}$$

Substituting Eq. (7) in Eq. (2) and using Eqns. (8)-(11), an equivalent differential equation on $\theta\in[0,\pi]$ is obtained that is collocated at N-3 collocation points given by Eq. (14) yielding N-4 equations in $N+1$ unknowns a_k. Imposing the boundary conditions, we get a system of four equations in $N+1$ unknowns for each of the boundary conditions given by Eqns. (3)-(5). The resulting $N+1$ by $N+1$ system of equations is expressed as a matrix eigenvalue problem and solved using a standard eigensolver. There is different eigenvalue solver in different software that can be utilized to solve the generalized eigenvalue problem. The algorithms used for computing the eigenvalues are the Cholesky factorization method or the QZ algorithm which is based on the generalized Schur decomposition. In general, the two algorithms return the result. The QZ algorithm is found to be stable in problems involving ill-conditioned matrices and is the main algorithm in the eigenvalue solver.

4. RESULTS AND DISCUSSIONS

With the plate having vertical simply supported edges, we denote the boundary conditions using two letters with the first letter denoting the bottom condition and the second letter denoting the top condition. For example, if the letters *C, S* and *F* are used to denote clamped, simply supported and free conditions then a plate with clamped bottom and simply supported top is denoted by *CS*. In all the cases, $\alpha=\frac{\pi}{a}(n=1)$ and the Poisson's ration υ is 0.3. The proposed PS method is first applied to obtain the fundamental frequency of vibration of a rectangular plate under self-weight. The algebraic eigenvalue problem obtained is solved for the eigenvalues using the eigensolver of MATLAB. The program was run for different values of N until we get the frequency parameter values correct to six decimal places. The results of the PS method obtained for $N=25$ (26 collocation points) are presented. The frequency parameter (K) for the *CF, CS* and SF plates are presented in Tables 1-3 respectively. The results obtained using the PS method are compared with the results obtained using the semi-analytic Levy-integration method [5]. In the tables $\gamma=0$ corresponds to the case when self-weight is absent.

The weight parameter γ is varied over the values 0, 7, 20 and 100 in Table 1; 0, 10, 50, 100, 200 in Table 2 and 0, 10, 50 and 100 in Table 3 with the width parameter a varying over the values 0.2, 0.5, 1 and 2 in Table 1, Table 2 and an additional value of $a=10$ in Table 3. The variation in the frequency parameter with variation in a and γ is reflected in the tables.

It is observed that for fixed width as the weight increases, the frequency decreases until the plate buckles statically. In the tables, the frequency parameter values marked "-'' means that the plate has already buckled. The results obtained using the PS method are almost same as those obtained using the semi analytic method. In the semi analytic method [5] the method of solution is tedious as two-point boundary value problem is first converted into two initial value problems and then the bisection algorithm was utilized to solve the resulting nonlinear equations. In the present work, the PS method efficiently obtains the frequency values with relatively good accuracy that is comparable with the results of reference [5]. In the second instance, the PS method is used to analyze the free transverse vibration of a rectangular plate resting on a Winkler foundation. Here the weight is absent and the foundation parameter (K_w) takes the values 100, 300 with the ratio (a) of width to height taking the values 1.0 and 2.0. The values of the frequency parameter (K^2) obtained using the PS method with $N=20$ (21 collocation points) are given in Table 4 for *SC* and *SS* plates. It is observed that the frequency parameter values are greater in *SC* plate case than in the *SF* plate case. Further it increases with the increasing value of the foundation modulus parameter and aspects ratio.

The results obtained are compared with those obtained using the differential transform method [15]. A comparison of the results obtained using the PS method show that the formulation can provide highly accurate results in a simple and efficient manner.

Table 1. Frequency parameter *K* for the *CF* plate

	γ							
	0		**7**		**20**		**100**	
a	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**
0.2	15.740	15.739725	15.739	15.739034	15.738	15.737751	15.729	15.729818
0.5	6.458	6.457703	6.448	6.447648	6.429	6.428792	6.306	6.306385
1	3.562	3.561932	3.501	3.501093	3.378	3.378442	1.388	1.387723
2	2.388	2.388276	2.158	2.158045	1.092	1.091664	-	-

Table 2. Frequency parameter *K* for the *CS* plate

	γ							
	0		**7**		**20**		**100**	
a	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**
0.2	16.048	16.048178	16.045	16.045315	16.045	16.033734	16.014	16.018945
0.5	7.189	7.188482	7.158	7.157805	7.027	7.026795	6.839	6.838520
1	4.863	4.862748	4.764	4.764093	4.265	4.264998	2.901	2.901079
2	4.163	4.163142	4.004	4.003524	2.933	2.932533	-	-
10	3.936	3.935991	3.745	3.745056	1.944	1.944364	-	-

Table 3. Frequency parameter *K* for the *SF* plate

	γ							
	0		**7**		**20**		**100**	
a	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**	**Ref. [5]**	**PS method**
0.2	15.735	15.734484	15.734	15.733494	15.730	15.729524	15.714	15.714396
0.5	6.419	6.418461	6.404	6.403488	6.341	6.340647	4.491	4.491206
1	3.148	3.418265	3.316	3.315907	2.623	2.623214	-	-
2	2.008	2.008405	0.958	0.958176	-	-	-	-

Table 4. The first three frequency parameter values (K^2) for plates on Winkler foundation

	***SC* Plate**				***SF* Plate**			
Kw	***a=1***		***a=2***		***a=1***		***a=2***	
	Ref. [15]	**PSM**	**Ref. [15]**	**PSM**	**Ref. [15]**	**PSM**	**Ref. [15]**	**PSM**
	25.6739	25.673886	52.6330	52.632980	15.3795	15.379479	42.3930	42.392972
100	59.4928	59.492822	86.7130	86.71300	29.5028	29.502790	59.9061	59.906048
	113.6690	113.668826	141.2000	141.200112	62.6637	62.663668	95.0114	9.011427
	29.3112	29.311233	54.4998	54.499822	20.8933	20.893262	44.6897	44.689641
300	61.1506	61.150600	87.8586	87.8586700	32.7172	32.717192	61.5527	61.552698
	114.5450	114.545196	141.9070	141.906559	64.2397	64.239671	96.0581	96.058166
500	32.5446	32.544561	56.3048	56.304801	25.2295	25.229514	46.8739	46.873917
	62.7646	62.764607	88.9896	88.989583	35.6429	35.642876	63.1564	63.15643
	115.4150	115.414913	142.6100	142.609507	65.7779	65.777924	97.0936	97.09362

5. CONCLUSION

The pseudospectral method is employed to analyze the vibration of a rectangular plate under self-weight and is also further used to analyze free transverse vibration of rectangular plates of uniform thickness resting on a Winkler foundation. The two opposite edges of the plate are assumed to be simply supported and different combinations of clamped, free and simply supported conditions are taken on the other two edges. The accuracy of the method is confirmed via comparison studies and the results obtained show the effectiveness of the method for free vibration studies.

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