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DISTANCE BETWEEN UNCERTAIN RANDOM VARIABLES

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ABSTRACT

In order to deal with indeterminacy data involving uncertainty and randomness, uncertain random variable is investigated by many scholars. As an extension of distance between uncertain variables, the definition of distance between uncertain random variables is proposed in this paper. Then, some formulas are provided to calculate distances between particular types of uncertain random variables.

Keywords: Uncertainty theory, Uncertain random variable, Chance measure, Distance.

1. INTRODUCTION

In order to deal with indeterminacy phenomenon in daily life, several axiomatic systems have been founded. In 1933, Kolmogorov [5] founded an axiomatic system of probability theory to describe random phenomena. If there are enough historical data, we can employ probability theory to estimate probability distribution. Sometimes, it is difficult to collect observed data when some unexpected events occur. In this case, people have to invite experts to estimate the belief degree of each event's occurrence. However, some counterintuitive consequences may occur if we employ probability theory or fuzzy set theory to model the belief degree [11]. For dealing with belief degree legitimately, an axiomatic system named uncertainty theory was proposed by Liu [7] in 2007. In addition, the product uncertain measure was defined by Liu [9] in 2009.

The concept of uncertain variable and uncertainty distribution were proposed by Liu [7]. Then, a sufficient and necessary condition of uncertainty distribution was proved by Peng and Imamura [18] in 2010. To describe the relationship between uncertain measure and uncertain distribution, a measure inversion theorem was presented by Liu [10] from which the uncertain measures of some events would be calculated via the uncertainty distribution. After proposing the concept of independence [9], Liu [10] presented the operational law of uncertain variables. The concepts of expected value, variance, moments and distance of uncertain variable were proposed by Liu [10]. Besides, a useful formula was presented by Liu and Ha [13] to calculate the expected values of monotone functions of uncertain variables. In order to characterize the uncertainty of uncertain variables, Liu [9] proposed the concept of entropy in 2009. After that, Dai and Chen citeDai12 proved the positive linearity of entropy and gave some formulas to calculate the entropy of monotone function of uncertain variables. Chen and Dai [1] discussed the method to select the uncertainty distribution using the maximum entropy principle. In order to make an extension of entropy, Chen, Kar and Ralescu [2] proposed a concept of cross-entropy for comparing an uncertainty distribution against a reference uncertainty distribution.

In 2013, Liu [14] proposed chance theory by defining uncertain random variable and chance measure in order to describe a system that involved both uncertainty and randomness. Some related concepts of uncertain random variables such as chance distribution, expected value, and variance were also presented by Liu [14]. As an important contribution to chance theory, Liu [15] proposed a basic operational method of uncertain random variables. After that, uncertain random variables were discussed widely. A law of large numbers was presented by Yao and Gao [23]. Besides, Yao and Gao [22] proposed an uncertain random process. As applications of chance theory, Liu [15] proposed uncertain random programming. Uncertain random risk analysis was presented by Liu [16]. Besides, chance theory was applied into many fields and many achievements were obtained, such as uncertain random reliability analysis (Wen and Kang [20]), uncertain random logic (Liu eLiuY13d), uncertain random graph (Liu [12]), and uncertain random network (Liu [12]).

In this paper, the distance between uncertain random variables is studied. In the first section, the uncertainty theory and uncertain random variables are introduced. In the following section, the definition of distance between uncertain random variables is presented and some formulas are proposed to calculate the distance between specific type's uncertain random variables. In addition, the method is illustrated by examples.

2. PRELIMINARY

As a branch of axiomatic mathematics, uncertainty theory aims to deal with human uncertainty. We will first present some basic concepts of uncertain theory and chance theory.

2.1 Uncertain variables

Suppose that Γ is a nonempty set and L is a σ-algebra on Γ. Each element in L is called an event. A set function M from L to [0,1] that satisfies normality axiom, duality axiom, subadditivity axiom (Liu [7]) and product axiom (Liu [9]) is called an uncertain measure. In general, $(\Gamma, \mathrm{L}, \mathrm{M})$ is called an uncertainty space. An uncertain variable ξ is defined as a measurable function from $(\Gamma, \mathrm{L}, \mathrm{M})$ to R. That is to say, the set $\xi^{-1}(B) = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event, for any Borel set B of real numbers.

Liu [7] presented the definition of uncertainty distribution Φ to represent uncertain variables, in which Φ is defined as $\Phi(x) = \mathrm{M}\{\xi \le x\}$ for any real number x.

Definition 1. (Liu [7]) If the uncertainty distribution of an uncertain variable ξ is

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x < a \\ (x-a)/(b-a), & \text{if} \quad a \le x < b \\ 1, & \text{if} \quad x \ge b, \end{cases}$$

Where a and b are real numbers with $a < b$, then ξ is called linear uncertain variable and its uncertainty distribution is denoted by $L(a,b)$.

Definition 2. (Liu [9]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables. If

$$\mathrm{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathrm{M}\{\xi_i \in B_i\}$$

For any Boral sets $B_1, B_2, \cdots, B_n$ of real numbers, then we say $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables.

Theorem 1. (Liu [7]) Let $f(x_1, x_2, \cdots, x_n)$ be strictly monotone increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly monotone decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent regular uncertain variables and $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$. Then, the inverse uncertainty distribution of ξ $\Psi^{-1}(\alpha)$ can be calculated by

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)),$$

In which $\Psi_1^{-1}(\alpha), \Psi_2^{-1}(\alpha), \cdots, \Psi_n^{-1}(\alpha)$ are inverse uncertainty distributions of $\xi_1, \xi_2, \cdots, \xi_n$, respectively.

In order to describe the mean value of an uncertain variable ξ by uncertain measure, Liu [7] defined the expected value of ξ as

$$E[\xi] = \int_0^{+\infty} \mathrm{M}\{\xi \ge r\} dr - \int_{-\infty}^{0} \mathrm{M}\{\xi \le r\} dr \tag{1}$$

Provided that at least one of the two integrals is finite. In addition, the expected value can be calculated by

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^{0} \Phi(x) dx. \tag{2}$$

Furthermore, the expected value of a function of n uncertain variables can be calculated by inverse uncertainty distributions.

Theorem 2. (Liu and Ha [13]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be n independent regular uncertain variables. If the conditions of Theorem 1 hold, then the expected value of uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) d\alpha. \tag{3}$$

Definition 3. (Liu [7]) Let ξ and η be two uncertain variables. The distance between ξ and η is defined as

$$d(\xi, \eta) = E[|\xi - \eta|] = \int_0^{+\infty} \mathrm{M}\{|\xi - \eta| \ge r\} dr. \tag{4}$$

Besides, the distance $d(\cdot,\cdot)$ satisfies no negativity, identification, symmetry and triangular inequality.

2.2 Uncertain random variables

In 2013, Liu [14] first proposed chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, operational law, expected value and so on. The chance space is refer to the product $(\Gamma, \mathrm{L}, \mathrm{M}) \times (\Omega, \mathrm{A}, Pr)$, in which $(\Gamma, \mathrm{L}, \mathrm{M})$ is an uncertain space and (Ω, A, Pr) is a probability space.

Definition 4. (Liu [14]) Let $(\Gamma, \mathrm{L}, \mathrm{M}) \times (\Omega, \mathrm{A}, Pr)$ be a chance space, and let $\Theta \in \mathrm{L} \times \mathrm{A}$ be an event. Then the chance measure of Θ is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega \mid \mathrm{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \ge r\} dr$$

Liu ([14]) proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

(a) $Ch\{\Gamma \times \Omega\} = 1$, $Ch\{\varnothing\} = 0$;

(b) $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$ For any event Θ;

(c) $Ch\{\Theta_1\} \le Ch\{\Theta_2\}$ For any event $\Theta_1 \subset \Theta_2$.

Lemma 1. (Hou ([4])) The chance measure is sub additive. That is, for any countable sequence of events $\Theta_1, \Theta_2, \cdots$, we have

$$Ch\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \le \sum_{i=1}^{\infty} Ch\{\Theta_i\}$$

Roughly speaking, an uncertain random variable is a measurable function of uncertain variables and random variables.

Definition 5. (Liu [14]) An uncertain random variable is a function ζ from a chance space $(\Gamma, \mathrm{L}, \mathrm{M}) \times (\Omega, \mathrm{A}, Pr)$ to the set of real numbers *i.e.*, $\{(\gamma,\omega) \in \Gamma \times \Omega \mid \zeta(\gamma,\omega) \in B\}$ is an event for any Boral set B.

Note that an uncertain random variable $\zeta(\gamma,\omega)$ is a bivariate function on $\Gamma \times \Omega$. Specifically, both random variables and uncertain variables are degenerated uncertain random variables.

The uncertain random arithmetic is defined as follows.

Definition 6. (Liu [14]) Let $f: R^n \to R$ be a measurable function, and $\zeta_1, \zeta_2, \cdots, \zeta_n$ be uncertain random variables on the chance space $(\Gamma, \mathrm{L}, \mathrm{M}) \times (\Omega, \mathrm{A}, Pr)$. Then, $\zeta = f(\zeta_1, \zeta_2, \cdots, \zeta_n)$ is an uncertain random variable defined as

$$\zeta(\gamma,\omega) = f(\zeta_1(\gamma,\omega), \zeta_2(\gamma,\omega), \cdots, \zeta_n(\gamma,\omega)),$$

For all $(\gamma,\omega) \in (\Gamma \times \Omega)$.

Definition 7. (Liu [14]) Let ζ be an uncertain random variable. The chance distribution of ζ is defined by $\Phi(x) = Ch\{\zeta \le x\}$ for any $x \in R$.

Theorem 3. (Liu [15]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, Upsilon_2, \cdots, \Upsilon_n$ respectively. Then the uncertain random variable

$$\zeta = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

Has a chance distribution

$$\Phi(x) = \int_{R^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m)$$

Where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$

And is determined by its inverse function

$$F^{-1}(\alpha; y_1, y_2, \cdots, y_m) = f(y_1, y_2, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots$$
$$, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha))$$

Provided that $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is a strictly increasing function with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing function with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$.

Definition8. (Liu [14]) Let ζ be an uncertain random variable. Then its expected value is defined by

$$E[\zeta] = \int_0^{+\infty} Ch\{\zeta \ge r\} dr - \int_{-\infty}^0 Ch\{\zeta \le r\} dr,$$

Provided that at least one of the two integrals is finite.

Definition9. (Liu [14]) Let ζ be an uncertain random variable with chance distribution Φ. If the expected value of ζ exists, then

$$E[\zeta] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx.$$

Theorem4. (Liu [14]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, Upsilon_2, \cdots, \Upsilon_n$ respectively. Then the uncertain random variable

$$\zeta = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

Has an expected value

$$E[\zeta] = \int_{R^m} \int_0^1 f(x, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha),$$
$$\Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)$$

Provided that $f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is a strictly increasing function with respect to $\tau_1, \tau_2, \cdots, \tau_k$ and strictly decreasing function with respect to $\tau_{k+1}, \tau_{k+2}, \cdots, \tau_n$.

3. DISTANCE BETWEEN UNCERTAIN RANDOM VARIABLES

Definition 1. The distance between uncertain random variables ζ_1 and ζ_2 is defined as

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|]. \tag{5}$$

That is,

$$d(\zeta_1, \zeta_2) = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \ge r\} dr.$$

Remark 1. If the uncertain random variables ζ_1 and ζ_2 de ζ_2 generate to uncertain variables, then

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|] = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \ge r\} dr = \int_0^{+\infty} \mathrm{M}\{|\zeta_1 - \zeta_2| \ge r\} dr.$$

It means that the definition of distance between uncertain random variables is consistent with uncertain variables.

Remark 2. If the uncertain random variables ζ_1 and ζ_2 degenerate to random variables, then

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|]$$
$$= \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \ge r\} dr = \int_0^{+\infty} Pr\{|\zeta_1 - \zeta_2| \ge r\} dr.$$

It means that the definition of distance between uncertain random variables is consistent with random variables.

Theorem 1. Let ζ_1, ζ_2 and ζ_3 be uncertain random variables, and let $d(\cdot,\cdot)$ be the distance. Then we have

(a)(No negativity) $d(\zeta_1,\zeta_2)\ge 0$;

(b)(Identification) $d(\zeta_1,\zeta_2)=0$ if and only if $\zeta_1=\zeta_2$;

(c)(Symmetry) $d(\zeta_1,\zeta_2)=d(\zeta_2,\zeta_1)$;

(d)(Triangle Inequality) $d(\zeta_1,\zeta_2)\le 2d(\zeta_1,\zeta_3)+2d(\zeta_2,\zeta_3)$.

Proof. The proofs of parts (a), (b) and (c) are trivial. Now we prove the part (d). By using the definition of distance and Lemma 2.1, we get

$$\begin{aligned}
&d(\zeta_1,\zeta_2)=E[|\zeta_1-\zeta_2|]=\int_0^{+\infty}Ch\{|\zeta_1-\zeta_2|\ge r\}dr\\
&\le\int_0^{+\infty}Ch\{|\zeta_1-\zeta_3|+|\zeta_2-\zeta_3|\ge r\}dr\\
&\le\int_0^{+\infty}Ch\{(|\zeta_1-\zeta_3|\ge r/2)\cup(|\zeta_2-\zeta_3|\ge r/2)\}dr\\
&\le\int_0^{+\infty}2Ch\{(|\zeta_1-\zeta_3|\ge r)\}+2Ch\{(|\zeta_2-\zeta_3|\ge r)\}dr\\
&=2E[|\zeta_1-\zeta_3|]+2E[|\zeta_2-\zeta_3|]=2d(\zeta_1,\zeta_3)+2d(\zeta_2,\zeta_3).
\end{aligned}$$

Example 1. Let $\Gamma=\{\gamma_1,\gamma_2,\gamma_3\}$. Define $\mathrm{M}\{\varnothing\}=0,\mathrm{M}\{\Gamma\}=1$ and $\mathrm{M}\{\Lambda\}=1/2$ for any subset Λ $(\Lambda\ne\varnothing,\Gamma)$. Let $\Omega=\{\omega_1,\omega_2\}$. Define $Pr\{\omega_1\}=1/3,Pr\{\omega_2\}=2/3$. We set uncertain random variables ζ_1 and ζ_2 as follows,

$$\zeta_1(\omega)=\begin{cases}\xi, & \text{if}\quad \omega=\omega_1\\ \eta, & \text{if}\quad \omega=\omega_2,\end{cases}\quad \zeta_2(\omega)=\begin{cases}\eta, & \text{if}\quad \omega=\omega_1\\ \xi, & \text{if}\quad \omega=\omega_2,\end{cases}\qquad \zeta_3=0,$$

In which

$$\xi(\gamma)=\begin{cases}1, & \text{if}\quad \gamma=\gamma_1\\ 1, & \text{if}\quad \gamma=\gamma_2\\ 0, & \text{if}\quad \gamma=\gamma_3,\end{cases}\quad \eta(\gamma)=\begin{cases}0, & \text{if}\quad \gamma=\gamma_1\\ -1, & \text{if}\quad \gamma=\gamma_2\\ -0, & \text{if}\quad \gamma=\gamma_3.\end{cases}$$

It is easy to verify that $d(\zeta_1,\zeta_2)=3/2$, $d(\zeta_1,\zeta_3)=1/2$ and $d(\zeta_2,\zeta_3)=1/2$. Thus

$$d(\zeta_1,\zeta_2)=\frac{3}{2}(d(\zeta_1,\zeta_3)+d(\zeta_2,\zeta_3)).$$

In the following, we will discuss the method to obtain distance from chance distributions. Let ζ_1 and ζ_2 be uncertain random variables with chance distributions $\Upsilon_1(x)$ and $\Upsilon_2(x)$, respectively. If $\zeta_1-\zeta_2$ has a chance distribution $\Upsilon(x)$, then the distance is

$$\begin{aligned}
&d(\zeta_1,\zeta_2)=\int_0^{+\infty}Ch\{|\zeta_1-\zeta_2|\ge x\}dx\\
&=\int_0^{+\infty}Ch\{(\zeta_1-\zeta_2\ge x)\cup(\zeta_1-\zeta_2\le -x)\}dx\\
&\le\int_0^{+\infty}(Ch\{\zeta_1-\zeta_2\ge x\}+Ch\{\zeta_1-\zeta_2\le -x\})dx\\
&\le\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx.
\end{aligned}$$

We stipulate that the distance between ζ_1 and is

$$d(\zeta_1,\zeta_2)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx. \tag{6}$$

Remark 3. Mention that (5) is not a precise formula but a stipulation. The calculation formula of distance between uncertain random variables in the rest of this paper is refer to (5).

Remark 4. Let η_1 and η_2 be random variables with probability distributions $\Psi_1(x)$ and $\Psi_2(x)$, respectively. If $\eta_1-\eta_2$ has a probability distribution $\Upsilon(x)$, then the distance between η_1 and η_2 is

$$\begin{aligned}
&d(\eta_1,\eta_2)=\int_0^{+\infty}Ch\{|\eta_1-\eta_2|\ge x\}dx\\
&=\int_0^{+\infty}Pr\{|\eta_1-\eta_2|\ge x\}dx\\
&=\int_0^{+\infty}Pr\{(\zeta_1-\zeta_2>x)\cup(\zeta_1-\zeta_2\le -x)\cup(\zeta_1-\zeta_2=x)\}dx\\
&=\int_0^{+\infty}(Pr\{\zeta_1-\zeta_2>x\}+Pr\{\zeta_1-\zeta_2\le -x\}+Pr\{\zeta_1-\zeta_2=x\})dx\\
&=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx
\end{aligned}$$

That means (5) is a precise formula when the uncertain random variables degenerate to random variables.

Theorem 2. Let $\eta_1,\eta_2,\cdots,\eta_m,\iota_1,\iota_2,\cdots,\iota_p$ be independent random variables with probability distributions $\Psi_1(x),\Psi_2(x),\cdots,\Psi_{m+p}$, and let $\tau_1,\tau_2,\cdots,\tau_n,\xi_1,\xi_2,\cdots,\xi_q$ be independent uncertain variables with uncertainty distributions $\Upsilon_1(x),\Upsilon_2(x),\cdots,\Upsilon_{n+q}(x)$ respectively. Let ζ_1 and ζ_2 be two uncertain random variables, in which

$$\zeta_1=f_1(\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n),$$

$$\zeta_2=f_2(\iota_1,\iota_2,\cdots,\iota_p,\xi_1,\xi_2,\cdots,\xi_q).$$

Then, the distance between ζ_1 and ζ_2 is

$$\begin{aligned}
&d(\zeta_1,\zeta_2)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx\\
&=\int_0^{+\infty}[1-\int_{R^{m+p}}F(x;y_1,\cdots,y_{m+p})\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_{m+p}(y_{m+p}) \qquad (7)\\
&+\int_{R^{m+p}}F(-x;y_1,\cdots,y_{m+p})\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_{m+p}(y_{m+p})]dx,
\end{aligned}$$

Where $F(x;y_1,y_2,\cdots,y_{m+p})$ is the uncertainty distribution of the uncertain variable

$$f_1(y_1,y_2,\cdots,y_m,\tau_1,\tau_2,\cdots,\tau_n)-f_2(y_{m+1},y_{m+2},\cdots,y_{m+p},\xi_1,\xi_2,\cdots,\xi_q)$$

And is determined by its inverse function

$$F^{-1}(\alpha;y_1,y_2,\cdots,y_{m+p})=f_1\left(y_1,y_2,\cdots,y_m,\Upsilon_1^{-1}(\alpha),\cdots,\Upsilon_k^{-1}(\alpha),\Upsilon_{k+1}^{-1}(1-\alpha),\cdots,\Upsilon_n^{-1}(1-\alpha)\right)$$
$$-f_2\left(y_{m+1},y_{m+2},\cdots,y_{m+p},\Upsilon_{n+1}^{-1}(1-\alpha),\cdots,\Upsilon_{n+t}^{-1}(1-\alpha),\Upsilon_{n+t+1}^{-1}(\alpha),\cdots,\Upsilon_{n+q}^{-1}(\alpha)\right)$$

provided that $f_1(\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n)$ is a strictly increasing function with respect to $\tau_1,\tau_2,\cdots,\tau_s$ and strictly decreasing function with respect to $\tau_{s+1},\cdots,\tau_n$ and $f_2(\iota_1,\iota_2,\cdots,\iota_p,\xi_1,\xi_2,\cdots,\xi_q)$ is a strictly increasing function with respect to $\xi_1,\xi_2,\cdots,\xi_t$ and strictly decreasing function with respect to $\xi_{t+1},\cdots,\xi_n$.

Proof. It follows from Theorem 3 immediately.

Corollary 1. Let η be a random variable with probability distribution $\Phi(x)$ and τ be an uncertain variable with uncertainty distribution $\Psi(x)$. Let $\Upsilon(x)$ be the chance distribution of $\tau-\eta$, and then we have

$$d(\eta,\tau)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx$$
$$=\int_0^{+\infty}\left(1-\int_{-\infty}^{+\infty}\Psi(x+y)d\Phi(y)+\int_{-\infty}^{+\infty}\Psi(-x+y)d\Phi(y)\right)dx$$

Proof. Note that $\Upsilon(x)=\int_{-\infty}^{+\infty}\Psi(x+y)d\Phi(y)$. It follows from Theorem 2 immediately.

Example 2. Let be a random variable with probability distribution $\Phi(x)$ and c be a real number with uncertainty distribution $\Psi(x)$. Suppose that $\Upsilon(x)$ represents the chance distribut η ion of $c-\eta$. According to Corollary 1, we have

$$\Upsilon(x)=\int_{-\infty}^{+\infty}\Psi(x+y)d\Phi(y)=\int_{c-x}^{+\infty}d\Phi(y)=1-\Phi(c-x),$$

In which

$$\Psi(x)=\begin{cases}1, & \text{if} \quad x\ge c\\ 0, & \text{if} \quad x<c.\end{cases}$$

Then we have

$$d(\eta,c)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx=\int_0^{+\infty}\left(1-\Phi(c+x)+\Phi(c-x)\right)dx$$

Example3. Let c be a real number with probability distribution $\Phi(x)$ and τ be an uncertain variable with uncertainty distribution $\Psi(x)$. Suppose that $\Upsilon(x)$ represents the chance distribution of $\tau-c$. According to Corollary 1, we have

$$\Upsilon(x)=\int_{-\infty}^{+\infty}\Psi(x+y)d\Phi(y)=\Psi(x+c),$$

In which

$$\Phi(x)=\begin{cases}1, & \text{if} \quad x\ge c\\ 0, & \text{if} \quad x<c.\end{cases}$$

Then we have

$$d(c,\tau)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx=\int_0^{+\infty}\left(1-\Psi(c+x)+\Psi(c-x)\right)dx$$

Example 4. Let b and c be two real numbers with distribution functions $\Psi(x)$ and $\Phi(x)$. Suppose that $\Upsilon(x)$ represents the chance distribution of $b-c$. According to Corollary 1, we have

$$\Upsilon(x)=\int_{-\infty}^{+\infty}\Psi(x+y)d\Phi(y)=\Psi(x+c)=\begin{cases}1, & \text{if} \quad x\ge b-c\\ 0, & \text{if} \quad x<b-c,\end{cases}$$

In which

$$\Phi(x)=\begin{cases}1, & \text{if} \quad x\ge c\\ 0, & \text{if} \quad x<c,\end{cases}\Psi(x)=\begin{cases}1, & \text{if} \quad x\ge b\\ 0, & \text{if} \quad x<b.\end{cases}$$

Then we have

$$d(c,b)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx=|b-c|$$

It means that the definition of distance between uncertain random variables is consistent with real numbers.

Corollary 2. Let η_1 and η_2 be two independent random variables with probability distributions $\Phi_1(x)$ and $\Phi_2(x)$, respectively. Let τ_1 and τ_2 be two independent uncertain variables with uncertainty distributions $\Psi_1(x)$ and $\Psi_2(x)$. Suppose that $\Upsilon(x)$ represents the chance distribution of $(\eta_1+\tau_1)-(\eta_2+\tau_2)$. Then we have

$$d(\eta_1+\tau_1,\eta_2+\tau_2)=\int_0^{+\infty}(1-\Upsilon(x)+\Upsilon(-x))dx$$
$$=\int_0^{+\infty}\left(1-\int_{R^2}F(x;y_1,y_2)d\Phi_1(y_1)d\Phi_2(y_2)+\int_{R^2}F(-x;y_1,y_2)d\Phi_1(y_1)d\Phi_2(y_2)\right)dx \tag{8}$$

Where $F^{-1}(\alpha;y_1,y_2)=y_1-y_2+\Psi_1^{-1}(\alpha)-\Psi_2^{-1}(1-\alpha)$ is the inverse distribution of the uncertain variable $y_1-y_2+\tau_1-\tau_2$.

Proof. It follows from Theorem 2 immediately.

Example 5. Let η be a random variable with probability distribution $\Phi(x)$ and τ be an uncertain variable with uncertainty distribution $\Psi(x)$, in which

$$\Phi(x)=\begin{cases}1-exp(-x), & \text{if} \quad x\ge 0\\ 0, & \text{if} \quad x<0,\end{cases}$$

$$\Psi(x)=\begin{cases}1, & \text{if} \quad x\ge 1\\ x, & \text{if} \quad 0\le x<1\\ 0, & \text{if} \quad x<0.\end{cases}$$

We have

$$\int_0^{+\infty} \Psi(x+y)d\Phi(y) = \begin{cases} x+1-exp(-1+x), & \text{if} \quad 0 \le x < 1 \\ 1, & \text{if} \quad x \ge 1, \end{cases}$$

And $\int_0^{+\infty} \Psi(-x+y)d\Phi(y) = exp(-x) - exp(-1-x), x \ge 0.$

According to Corollary 1, we have

$$\begin{aligned} d(\eta,\tau) &= \int_0^{+\infty} \left(1 - \int_{-\infty}^{+\infty} \Psi(x+y)d\Phi(y) + \int_{-\infty}^{+\infty} \Psi(-x+y)d\Phi(y)\right)dx \\ &= \int_0^1 \left(-2x + exp(-1+x) + 1 - exp(-x-1)\right)dx + \int_1^{+\infty} \left(exp(-x) - exp(-1-x)\right)dx \\ &= 1/2 + exp(-1) - exp(-1)^2 = 0.7325. \end{aligned}$$

4. CONCLUSIONS

In this paper, the concept of distance between uncertain variables was expanded to uncertain random variables. Based on the subadditivity of chance measure and expected value of uncertain random variables, several properties of the distance were proved. Then, the effectiveness of this method was illustrated by an example.

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