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Existence of Resonance Stability of Triangular Equilibrium Points for an Oblate Infinitesimal in Elliptical Restricted Three Body Problem with Radiating Oblate Primaries

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Abstract

The present paper models the restricted three body problem, considering the generalization that the orbits of the primaries are taken to be elliptic and the two primaries are considered to be sources of radiation and all three participating bodies are considered as oblate spheroids. Hamiltonian of the problema is derived and then normalized using well-established normalization techniques. The range of values of μ and e for the linear stability of triangular equilibrium points have been found in presence of resonance. The stability of some of the cases of third order resonances has been simulated and explored graphically. The linear stability is observed in the resonance cases $3\lambda_2 = -1$, $3\lambda_2 = -2$ and $\lambda_1 + 2\lambda_2 = 0$, where as the triangular points are found to be linearly unstable in the case $\lambda_1 - 2\lambda_2 = 2$.

Key words

Elliptical restricted three body problem, Photogravitational, Dynamical system, Resonance, Linear stability.

1. Introduction

The photo-gravitational elliptical restricted three-body problem deals with the motion of a passively gravitating infinitesimal particle, which in addition to the gravitational force, is affected by the repulsive force of the light pressure from one or two primary bodies. The primaries are

assumed to be moving in elliptical orbit. This problem is obtained as a generalization of the Circular Restricted Three Body Problem (CRTBP), where the eccentricity of the orbits of the primaries is assumed to be greater than zero. The CRTBP with the effect of the radiation pressure, when one or both the primaries of the system are the source of radiation, was discussed by Radzievskii [25, 26].

The rotation of celestial bodies produces an equatorial bulge, which results in the oblateness of the body. In classical problems the primaries are taken strictly as spheres but some planets and stars are sufficiently oblate to make departure from sphericity significant in the study of celestial systems. The influence of eccentricity of the orbit of the primaries with or without radiation pressure, oblateness and triaxiality of the primaries was studied by many authors [1,3,7,13,15,16,18,20,21,24,29,34,36] and others. Kumar and Ishwar [16] investigated the stability of the collinear liberation points when both the primaries are oblate and radiating whereas Singh and Aishetu [30] studied the stability of the triangular equilibrium points when both the primaries are oblate and radiating. Narayan and Singh [22, 23] studied the motion and stability of triangular equilibrium points when the primaries are radiating.

The restricted three body problem when the oblateness of the infinitesimal is considered was also studied by some authors [4, 5]. Singh and Haruna [28] investigated the problem considering all the three participating bodies as oblate spheroid and reported the presence of five collinear equilibrium points. Also, they examined the stability of all the planar equilibrium points. [31] studied the dynamics of the planar ERTBP considering the oblateness of all three participating bodies and applied the model to binary pulsars.

In the study of phenomenon of resonance in the dynamics of solar system, Roy and Ovenden [27] established that among the planetary and satellite systems, the occurrence of commensurability between the pairs of mean motions is more frequent than in a chance distribution. The existence of a mean motion resonance between a pair of objects can lead to a repeating geometrical configuration of the orbits which guarantees stability even if the resonance is not exact, since there is still the possibility of stable liberal motion around an equilibrium point. Therefore, it is important to have an understanding of the dynamics of resonance. Since the late twentieth century until today, the enormous number of researches has enriched the study of Restricted Three-Body Problem (RTBP), by considering the influence of the various perturbing forces such as eccentricity of orbits, forces due to radiation pressure and oblateness. Markeev [19] studied the stability of equilibrium points in the presence of resonance greater than equal to 3 for the Hamiltonian system of an infinitesimal moving under the influence of two large gravitating bodies. Kumar and Choudhary [14] generalized the results given by Markeev by considering

doubly photogravitational elliptic restricted three body problem. Ferraz-Mello [6] derived a completely integrable dynamical system that represents the averaged motion of an asteroid moving in a first-order resonance with Jupiter. Hadjidemetriou [9] considered the resonant structure of the restricted three body problem for the Sun- Jupiter asteroid system in the plane is studied, both for a circular and an elliptic orbit of Jupiter and studied three typical resonance cases, the 2:1, 3:1 and 4:1 mean motion resonance of the asteroid with Jupiter. The resonance cases of libration points for restricted/elliptic restricted three body problem was analyzed by many authors: Henrard and Caranicolas [10–12], Hadjidemetriou [8], SubbaRao and Sharma [32], Thakur and Singh [33], Beauge´ etal. [2], Usha and Narayan[35] and many others.

In this paper, we attempt to present a generalized result for the restricted three body problem of linear stability around triangular equilibrium point taking into consideration third and fourth order resonances. Our results are generalized in the sense that the eccentricity of the orbit is considered to be non-zero and oblateness of all three participating bodies are taken into consideration. Also the two primaries are assumed to be radiating bodies.

The present paper is organized as follows: Section 1, presents a brief introduction. In Section 2 the equations of motion are presented and triangular equilibrium points are obtained. Section 3 focuses on Characteristic Roots and First order stability for the case when $e = 0$. In section 4, various canonical transformations are employed to obtain the normalized Hamiltonian for the system; in this section we follow [15]. In section 5, a study of resonance cases is presented. The discussions and conclusions are drawn in Section 6.

2. Equation of Motion

Assume that m_1, m_2 and m are the masses of the bigger, smaller and infinitesimal bodies respectively, where m_1 and m_2 have elliptical orbits and m is moving under their gravitational effect but the mass m being too small does not affect the motion of the primaries.

Let A_1, A_2 and A_3 denote the oblateness factor of the bigger primary, smaller primary and infinitesimal respectively. Also $q_1$and q_2 are assumed to be mass reduction factors of the two primaries. The frame of reference is so chosen that the distance between the primaries and gravitational constant are unity. Also the sum of the masses of the primaries is taken to be unity and mass ratio is given as $\mu = \frac{m_2}{m_1+m_2}$. Thus, position offirst primary is $(\mu, 0, 0)$ and second primary is $(\mu - 1, 0, 0)$. Then the equation of motion in the pulsating rotating barycentric frame of reference is given as:

$$x'' - 2y' = \frac{1}{1+e\cos f} U_x^*;$$
$$y'' + 2x' = \frac{1}{1+e\cos f} U_y^*; \quad (1)$$
$$z'' = \frac{1}{1+e\cos f} U_z^*;$$

where,

$$U^* = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[(1-\mu)\left(\frac{q_1}{r_1} + \frac{q_1A_1+A_3}{2r_1^3}\right) + \mu\left(\frac{q_2}{r_2} + \frac{q_2A_2+A_3}{2r_2^3}\right)\right]. \quad (2)$$

Here, f denotes the true anomaly of one of the primaries and (‘) denotes differentiation with respect to f. Since the motion of the primaries are not affected by the infinitesimal body, the mean motion n [29] is given by

$$n^2 = \frac{1}{a^3}\left(1 + \frac{3}{2}(e^2 + A_1 + A_2)\right). \quad (3)$$

The perturbed Hamiltonian of the dynamical system described by the equations of motion given by the system (1) is presented as follows:

$$H = -\frac{x'^2+y'^2}{2} - (y'x - x'y) + P_x^2 + P_y^2 + P_x y - P_y x \frac{1}{1+e\cos f}[(1-\mu)(\frac{r_1^2}{r_1} + \frac{1}{n^2 r_1}(q_1 + \frac{q_1A_1+A_3}{2r_1^2})) + \mu(\frac{r_2^2}{2} + \frac{1}{n^2 r_2}\left(q_2 + \frac{q_2A_2+A_3}{2r_2^2}\right) \quad (4)$$

where, P_x and P_y denotes the generalized components of the momentum. The triangular equilibrium points in the case of planar three body problem is obtained by solving the equation, $H_x = 0,\ H_y = H_{px} = H_{py} = 0$ for $' = x'' = y' = y'' = 0 = z$. The triangular points given as $((x^*, \pm y^*, \pm p_x^*,\ \pm p_y^*)$ in linear trems of all the perturbing factors is given as:

$$x^* = \frac{1}{2} - \mu + \frac{\beta_2}{3} - \frac{\beta_1}{3} + \frac{A_1}{2} - \frac{A_2}{2} - A_3\ ,$$
$$y^* = \frac{\sqrt{3}}{2}\left(1 - \frac{2}{3}e^2 - \frac{5}{3}\alpha - \frac{2}{9}\beta_1 - \frac{2}{9}\beta_2 - \frac{A_1}{3} - \frac{A_2}{3}\right), \quad (5)$$
$$p_x^* = \frac{\sqrt{3}}{2}\left(1 - \frac{2}{3}e^2 - \frac{5}{3}\alpha - \frac{2}{9}\beta_1 - \frac{2}{9}\beta_2 - \frac{A_1}{3} - \frac{A_2}{3}\right),$$
$$p_y^* = \frac{1}{2} - \mu + \frac{\beta_2}{3} - \frac{\beta_1}{3} + \frac{A_1}{2} - \frac{A_2}{2} - A_3.$$

Here, $a = 1 - \alpha$.

3. Characteristic Roots and First Order Stability

In the further analysis, the nature of motion near the triangular point L_4 is studied, as L_5 *is* symmetrical to L_4. Assuming (qi, pi), i = 1, 2 to be the variation in the coordinates of the triangular point L_4, the variational equations may be written as:

$$\frac{dq_i}{df} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{df} = -\frac{\partial H}{\partial q_i}; (i = 1,2), \tag{6}$$

where,

$$H = H_0 + H_1 + H_2 + \cdots, \tag{7}$$

$$H_0 = cont,\ H_1 = 0, \tag{8}$$

$$H_2 = H_2^{(0)} + H_2^{(1)} \tag{9}$$

For H_2, the two parts are given as follows:

$$H_2^{(0)} = p_1 q_2 - p_2 q_1 + \frac{p_1^2 + p_2^2}{2} + [(\frac{1}{8} + A^{(0)}) q_1^2 - (K^{(0)} - B^{(0)}) q_1 q_2 - (\frac{5}{8} + C^{(0)}) q_2^2] \tag{10}$$

and

$$H_2^{(1)} = \frac{e \cos f}{1 + e \cos f} [(\frac{3}{8} - A) q_1^2 + (K - B)\, q_1 q_2 + (\frac{9}{8} + C)\, q_2^2] \tag{11}$$

Here, $H_2^{(0)}$ denotes the value of the second order Hamiltonian when eccentricity e is assumed to be 0 and the values of the coefficients are as follows:

$$A = \frac{7e^2}{8}(1 - 2\mu) + \frac{\beta_1}{4}(1 - 3\mu) - \frac{\beta_2}{4}(2 - 3\mu) - \frac{3A_1}{4}(1 - \frac{7}{4}\mu) + \frac{3A_2}{4}(1 - \frac{7}{4}\mu) + \frac{\alpha}{32} - 140 + \frac{9A_3}{8}(1 - \frac{7}{3}\mu),$$

$$B = \sqrt{3}[-\frac{11}{4}e^2(1 - \frac{20}{11}\mu) + \frac{\beta_1}{6}(1 + \mu) - \frac{\beta_2}{6}(2 - \mu) - \frac{5\alpha}{3}(1 - 2\mu) - \frac{A_1}{2}(7 - \frac{59}{4}\mu) - A_2(2 - \frac{29}{8}\mu) - \frac{9A_3}{4}(1 - \frac{5}{3}\mu) - \frac{29\alpha}{16}(1 - \frac{38}{29}\mu)]$$

$$K = \frac{3\sqrt{3}}{4}(1 - 2\mu) \tag{12}$$

$$C = \frac{e^2}{8}(23 - 22\mu) + \frac{\beta_1}{4}(1 - 3\mu) - \frac{\beta_2}{4}(2 - 3\mu) + \frac{3A_1}{4}(3 + \frac{11}{4}\mu) + \frac{3A_2}{4}(3 - \frac{11}{4}\mu) - \frac{\alpha}{32}(95 - 220\mu) - \frac{33A_3}{8}(1 - \mu)$$

The values of coefficients $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ and $K^{(0)}$ are obtained from A, B, C and K by substituting e=0.The canonical transformation for the second order Hamiltonian H_2 isgivenby

$$\frac{dq_i}{dt} = \frac{\partial H_2}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H_2}{\partial q_i}; (i = 1,2) \tag{13}$$

where, H_2 is given by equation (9). Thus the canonical equation of motion is obtained as:

$$\ddot{q}_1 - 2\dot{q}_2 = A^* q_1 + B^* q_2 ;$$
$$\ddot{q}_2 + 2\dot{q}_1 = B^* q_1 + C^* q_2 ; \tag{14}$$

where,

$$A^* = \frac{3}{4} - 2A,$$
$$B^* = K - B,$$
$$C^* = \frac{9}{4} + 2C.$$

Here, (‘)denotes differentiation w.r.t time t. Assuming that the solution for the system of equations given by(14) are $q_1=Le^{\lambda t}$ and $q_2=Me^{\lambda t}$, the characteristic equation is obtained as:

$$\lambda^4 + (4 - A^* - C^*)\lambda^2 + A^* C^* - B^{*2} = 0 \tag{15}$$

The equilibrium position is stable, if the roots of equation (15) are purely imaginary, thus solving the obtained condition for the case when both the frequencies are equal, we get the value of μ admisible for stable equilibrium point denoted by $\mu^{(e)}$ as:

$$\mu^{(e)} = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) + \frac{85\alpha}{9\sqrt{69}} + \frac{e^2}{9\sqrt{69}}\left(22 - \left(\frac{703\sqrt{69}}{4} - \frac{10775}{23}\right)\alpha\right) - \frac{1}{9}\left(1 - \sqrt{\frac{23}{3}} + \frac{1}{828}\left(83283 - 7748\sqrt{69}\right)\alpha\right)A_1 - \frac{1}{9}\left(4 + \frac{11}{\sqrt{69}} - \left(\frac{665}{12} - \frac{67}{6\sqrt{69}}\right)\alpha\right)A_2 + \frac{1}{3}\left(\frac{5}{6} + \frac{4}{\sqrt{69}} + \left(\frac{679\,3}{4\sqrt{69}} - \frac{3605}{36}\right)\alpha\right)A_3 + \frac{1}{9}\left(\frac{5}{\sqrt{69}} - \frac{35}{276}\left(529 + 4\sqrt{69}\right)\alpha\right)\beta_1 - \frac{1}{27}\left(\frac{2}{\sqrt{69}} + \left(\frac{51}{2} - \frac{3977}{12\sqrt{69}}\right)\alpha\right)\beta_2 \tag{16}$$

And the value of μ admisible for stable equilibrium point for the case when e=0, denoted by $\mu^{(0)}$ is given as

$$\mu^{(0)} = 0.0385209 - 0.419121\alpha - 0.795884A_1 - 0.407974A_2 - 0.682187A_3 - 0.00891747\beta_1 \quad - 0.00891747\beta_2 \tag{17}$$

4. Normalization of the Hamiltonian function H_2

In this section we shall study the stability of the triangular points in elliptical restricted three body problema adopting the method given by Markeev [19]. For subsequent studies, we first normalize the Hamiltonian upto second order given by (9). For normalization, we consider the canonical transformation

$$(q_1, q_2, p_1, p_2) = \left(q'_1, q'_2, p'_1, p'_2\right), \tag{18}$$

where ,

$$N = \begin{bmatrix} a_1 & a_1c_1 & -a_1c_1 & a_1(1-\omega_1^2 b_1) \\ a_2 & a_2c_2 & -a_2c_2 & a_2(1-\omega_2^2 b_2) \\ 0 & a_1b_1 & a_1(1-b_1) & a_1c_1 \\ 0 & -a_2b_2 & a_2(1-b_2) & -a_2c_2 \end{bmatrix},$$

$$a_i = \frac{1}{2}\left(\frac{2l_i}{\omega_i^2 - \frac{1}{2}}\right)^{1/2}, \tag{19}$$

$$b_i = \frac{1}{l_i},$$

$$c_i = \frac{-(K-B)}{l_i},$$

and

$$l_i = \frac{9}{4} + 2c + {\omega_i}^2.$$

Assuming the frequencies ω_1 and ω_2, are given by the relation $\omega_1^2 = -\{\lambda_{1,2}^{(0)}\}^2$ and $\omega_2^2 = -\{\lambda_{3,4}^{(0)}\}^2$ and the values are obtained as:

$$(\omega_{1,2})^2 = \frac{1}{2}[1 \pm \{1 - 27\mu(1-\mu)(1 + \frac{2}{9}\beta_1 + \frac{2}{9}\beta_2 + \frac{94}{9}\alpha + \frac{119}{6}A_1 + \frac{61}{6}A_2 + 17A_3)\}^{1/2} \times (1 - \frac{13}{4}\alpha - 6A_1 - 3A_2 - 6A_3)]. \tag{20}$$

The transformation (18) reduces the Hamiltonian (9) to the form

$$H'_2 = \frac{1}{2}\left(p'^2_1 + w_1^2 q'^2_1\right) - \frac{1}{2}\left(p'^2_2 + w_2^2 q'^2_2\right) + \frac{e\cos f}{1+e\cos f}\sum_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2=2} a'_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2}\, q_1'^{\upsilon_1} q_2'^{\upsilon_2} p_1'^{\gamma_1} p_2'^{\gamma_2}; \tag{21}$$

where,

$$a'_{2000} = ((\tfrac{3}{8} - A) + (K - B)c_1 \ + (\tfrac{9}{8} + \mathrm{C})c_1^2)a_{1,}^2$$

$$a'_{0200} = \left(\left(\tfrac{3}{8} - A\right) + (K - B)c_2 \ + \left(\tfrac{9}{8} + \mathrm{C}\right)c_2^2\right)a_2^2,$$

$$a'_{0020} = \left(\tfrac{9}{8} + \mathrm{C}\right)a_1^2\, b_1^2,$$

$$a'_{0002} = \left(\tfrac{9}{8} + \mathrm{C}\right)a_2^2\, b_2^2,$$

$$a'_{1100} = \left(\left(\tfrac{3}{4} - 2A\right) + (K - B)(c_1 + c_2) + \left(\tfrac{9}{4} + 2\mathrm{C}\right)c_1c_2\ \right)a_1a_2, \tag{22}$$

$$a'_{1010} = \left((K - B) + \left(\tfrac{9}{8} + 2\mathrm{C}\right)c_1\right)a_1^2\, b_1,$$

$$a'_{1001} = -\left((K - B) + \left(\tfrac{9}{8} + 2\mathrm{C}\right)c_1\right)a_1a_2b_2,$$

$$a'_{1010} = -\left((K - B) + \left(\tfrac{9}{8} + 2\mathrm{C}\right)c_2\right)a_2^2\, b_2,$$

$$a'_{0110} = ((K - B) \ - (\tfrac{9}{8} + 2\mathrm{C})c_1)a_1a_2b_{1,}$$

$$a'_{0011} = \ -(\tfrac{9}{8} + 2\mathrm{C})a_1a_2b_1b_2.$$

Next we apply the transformation

$$(q'_1, q'_2, p'_1, p'_2) = (\tfrac{1}{\sqrt{\omega_1}}\tilde{q}_1, \tfrac{1}{\sqrt{\omega_2}}\tilde{q}_2, \sqrt{\omega_1}\ \tilde{p}_1, \sqrt{\omega_2}\ \tilde{p}_2) \tag{23}$$

Consequently, we obtain the Hamiltonian in the form

$$\tilde{H}_2 = \frac{1}{2}w_1\left(\tilde{p}_1{}^2 + \tilde{q}_1{}^2\right) - \frac{1}{2}w_2\left(\tilde{p}_2{}^2 + \tilde{q}_2{}^2\right)$$

$$+\frac{e\cos f}{1+\cos f}\sum\nolimits_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2=2}\breve{a}_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2}\ \tilde{q}_1{}^{\upsilon_1}\tilde{q}_2{}^{\upsilon_2}\ \tilde{p}_1{}^{\gamma_1}\tilde{p}_2{}^{\gamma_2}; \tag{24}$$

$$\tilde{a}_{2000} = \frac{1}{w_1}a'_{2000,}\tilde{a}_{0200} = \frac{1}{w_2}a'_{0200,}\tilde{a}_{0200} = w_1a'_{0020,}$$

$$\tilde{a}_{0002} = w_2a'_{0002,}\tilde{a}_{1100} = \frac{1}{\sqrt{w_1w_2}}a'_{1100,}\tilde{a}_{1001} = \sqrt{\frac{w_2}{w_1}}a'_{1001,}\tilde{a}_{1010} = a'_{1010,} \tag{25}$$

$$\tilde{a}_{0110} = \sqrt{\frac{w_1}{w_2}}a'_{0110,}\tilde{a}_{0101} = a'_{0101,}\tilde{a}_{0011} = \sqrt{w_1w_2}a'_{0011}\,.$$

The Hamiltonian given by (24) is reduced to the form H_2''=$2iH_2$ by using the complex conjugate variable given by

$$q''_j = \tilde{p}_j + i\tilde{q}_j \ , p''_j = \tilde{p}_j - i\tilde{q}_j, \ \ (j = 1,2). \tag{26}$$

Thus Hamiltonian H_2'' is given as:

$$H_2 = iw_1 p''_1 q''_1 - iw_2 p''_2 q''_2 \tag{27}$$
$$+2i\frac{e\cos f}{1+\cos f}\sum_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2=2} a''_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2}\, q''^{\upsilon_1}_1 q''^{\upsilon_2}_2\, p''^{\gamma_1}_1 p''^{\gamma_2}_2$$

where,

$$a''_{2000} = \frac{1}{4}(-\tilde{a}_{2000} + \tilde{a}_{0020} - i\tilde{a}_{1010}),$$
$$a''_{0200} = \frac{1}{4}(-\tilde{a}_{0200} + \tilde{a}_{0002} - i\tilde{a}_{0101}),$$
$$a''_{1100} = \frac{1}{4}(-\tilde{a}_{1100} + \tilde{a}_{0011} - i\tilde{a}_{1001} - i\tilde{a}_{0110}), \tag{28}$$
$$a''_{1001} = \frac{1}{4}(\tilde{a}_{1100} + \tilde{a}_{0011} + i\tilde{a}_{1001} - i\tilde{a}_{0110}),$$
$$a''_{1010} = \frac{1}{2}(\tilde{a}_{2000} + \tilde{a}_{0020}),$$
$$a''_{0101} = \frac{1}{2}(\tilde{a}_{0200} + \tilde{a}_{0002}),$$

And other coefficient can be obtained from these coefficients as for the Hamiltonian $H''_2, a''_{\upsilon_1\upsilon_1\gamma_1\gamma_2} = \bar{a}''_{\gamma_1\gamma_2\upsilon_1\upsilon_1}$, where the bar sign denotes the complex conjúgate quantity. To reduce the Hamiltonian given by (27) to the normal form in complex conjúgate variables, the following transformation is applied

$$(q''_j, p''_j) \to (q_j^{**}, p_j^{**}), \tag{29}$$

given by the generating function:

$$q''_1 p_1^{**} + p''_2 q_2^{**} + S\left(q''_1 . q''_2 . q''_1 . p_1^{**} p_2^{**}, f\right), \tag{30}$$

where

$$S = \sum_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2=2} S_{\upsilon_1\upsilon_1\gamma_1\gamma_2}\, q''^{\upsilon_1}_1 q''^{\upsilon_2}_2\, p_1^{**\gamma_1} p_2^{**\gamma_2}. \tag{31}$$

And $S_{\upsilon_1\upsilon_1\gamma_1\gamma_2}$ are to be chosen 2π- periodic function of f . So that the Hamiltonian is in the form

:

$$H_2^*\left(q_j^{**}, p_j^{**}\right) = i\lambda_1 q_1^{**} p_1^{**} + i\lambda_2 q_2^{**} p_2^{**}. \tag{32}$$

Assuming the relation between the variables $q_j^{"}, p_j^{"}$ and q_j^{**}, p_j^{**} as:

$$q_j^{**} = q_j^{"} + \frac{\partial S}{\partial p_j^{**}},$$

$$p_j^{"} = p_j^{**} + \frac{\partial S}{\partial q_j^{**}} \tag{33}$$

We get the identity $H_2^{**}(q_j^{"} + \frac{\partial S}{\partial p_j^{**}}.p_j^{**}, f) - H_2^{"}(q_j^{"}, p_j^{**} + \frac{\partial S}{\partial q_j^{"}}.f) = \frac{\partial S}{\partial f}$.

On expanding using the Taylor's theorem upto second order derivative terms, we get:

$$H_2^{**}(q_j^{"} p_j^{**}, f) + \sum_{j=1}^{2} \frac{\partial S}{\partial p_j^{**}} \frac{\partial H_2^{**}}{\partial q_j^{"}} + \frac{1}{2}[(\frac{\partial S}{\partial p_1^{**}})^2 \frac{\partial^2 H_2^{**}}{\partial q_1^{"2}} + 2\frac{\partial S}{\partial p_1^{**}} \frac{\partial S}{\partial p_2^{**}} \frac{\partial^2 H_2^{**}}{\partial q_1^{"} \partial q_2^{"}} + (\frac{\partial S}{\partial p_8^{**}})^2 \frac{\partial^2 H_2^{**}}{\partial q_2^{"2}}]$$

$$-H_2^{"}((q_j^{"} p_j^{**}, f) - \sum_{j=1}^{2} \frac{\partial S}{\partial p_j^{"}} \frac{\partial H_2^{"}}{\partial p_j^{**}} - \frac{1}{2}[(\frac{\partial S}{\partial q_1^{"}})^2 \frac{\partial^2 H_2^{"}}{\partial p_1^{**2}} + 2\frac{\partial S}{\partial q_1^{"}} \frac{\partial S}{\partial q_2^{"}} \frac{\partial^2 H_2^{**}}{\partial p_1^{**} \partial p_2^{**}} + (\frac{\partial S}{\partial q_2^{"}})^2 \frac{\partial^2 H_2^{**}}{\partial p_2^{**2}}]$$

$$= \sum_{\upsilon_1+\upsilon_1+\gamma_1+\gamma_2=2} \frac{dS_{\upsilon_1\upsilon_1\gamma_1\gamma_2}}{df} q"_1{}^{\upsilon_1} q"_2{}^{\upsilon_2}\, p_1{}^{"\gamma_1} p_2{}^{"\gamma_2} \tag{34}$$

Using the equations (27) and (32), restricting only upto the second order terms in e, the above equation (34) is simplified in the form:

$$i\lambda_1 q_1^{"} p_1^{**} + i\lambda_2 q_2^{"} p_2^{**} + i\sum(\gamma_1\,\lambda_1 + \gamma_2\lambda_2)(es^{(1)} + e^2 s^{(2)}) - iw_1 q_1^{"} p_1^{**} + iw_2 q_2^{"} p_2^{**}$$

$$-2i[e\cos f - \frac{e^2}{2}(1 + \cos 2f)]\; a_{\upsilon_1\upsilon_1\gamma_1\gamma_2}^{"} q"_1{}^{\upsilon_1} q"_2{}^{\upsilon_2}\, p_1{}^{**\gamma_1} p_2{}^{**\gamma_2}$$

$$-i\sum(\upsilon_1 w_1 - \upsilon_2 w_2)(es^{(1)} + e^2 s^{(2)}) = e\frac{ds^{(1)}}{df} + e^2\frac{ds^{(2)}}{df} \tag{35}$$

where ,

$$S_{\upsilon_1\upsilon_1\gamma_1\gamma_2} = e\sum s_{\upsilon_1\upsilon_2\gamma_1\gamma_2}^{(1)} + e^2\sum s_{\upsilon_1\upsilon_2\gamma_1\gamma_2}^{(1)}$$

Assuming $\lambda_j = \lambda_j^{(0)} + e\lambda_j^{(1)} + e^2\lambda_j^{(2)} + \cdots, (j = 1,2)$ and equating the coefficients of the equal powers in e and integrating w.r.t f, we get

$$\lambda_1^{(0)} = \omega_1, \lambda_2^{(0)} = \omega_2, \tag{36}$$

$$s_{1010}^{(1)} = i\lambda_1^{(1)} f - 2ia_{1010}^{''} \sin f\ ,$$

$$s_{0101}^{(1)} = i\lambda_2^{(1)} f - 2ia_{0101}^{''} \sin f\ ,$$

$$s_{\upsilon_1\upsilon_1\gamma_1\gamma_2}^{(1)} = \frac{2ia_{\upsilon_1\upsilon_1\gamma_1\gamma_2}^{''}[\sin f + i\{(\upsilon_1-\gamma_1)w_1-(\upsilon_2-\gamma_2)w_2\}\cos f]}{\{(\upsilon_1-\gamma_1)w_1-(\upsilon_2-\gamma_2)w_2\}^2-1} \tag{37}$$

By virtue of periodicity of $s_{1010}^{(1)}$ and $s_{0101}^{(1)}$ it follows that $\lambda_1^{(1)} = \lambda_2^{(1)} = 0.$ Using relations given by (37), the value of S as a complex –valued function in the first order terms of e is obtained.

Again the Hamiltonian given by equation (27) is transformed to the normal form given by :

$$H_2^* = \frac{1}{2}\lambda_1(q_1^{*2} + p_1^{*2}) + \frac{1}{2}\lambda_2(q_2^{*2} + p_2^{*2}), \tag{38}$$

where the transformation is given by means of generating function

$\tilde{q}_1 p_1^* + \tilde{q}_2 p_2^* + K(\tilde{q}_j, p_j^*$,f), where K is restricted to the order of e alone and the relation between the variables is given as :

$$q_j^* = \tilde{q}_j + \frac{\partial K}{\partial p_j^*}, \tilde{p}_j = p_j^* + \frac{\partial K}{\partial q_j^*} \tag{39}$$

Taking into account the relation between the complex canonical variables with the real ones given by equation (26) and as follows :

$$q_j^{**} = p_j^* + iq_j^*, \quad p_j^{**} = p_j^* - iq_j^* (j = 1,2), \tag{40}$$

From relation (33),where S is taken to the order of e and given as $s^{(1)}(p_j^* + q_j^*, p_j^* - iq_j^*, f)$ is denoted by W(p_j^*, q_j^*, f), we obtain:

$$\tilde{q}_j = q_j^* - \frac{1}{2i}\frac{\partial W}{\partial p_j^*}, \tilde{p}_j = p_j^* + \frac{1}{2i}\frac{\partial K}{\partial q_j^*} \tag{41}$$

Comparing relation (39) and (41),we obtain:

$$K = \frac{1}{2} W \tag{42}$$

Hence the function $K = \sum k_{\upsilon_1 \upsilon_2 \gamma_1 \gamma_2} q_1^{*\upsilon_1} q_2^{*\upsilon_2} p_1^{*\gamma_1} p_2^{*\gamma_2}$ is revaluated and using the formula (41), (42) and (26), we get the coefficients $k_{\upsilon_1 \upsilon_2 \gamma_1 \gamma_2}$ as follows:

$$\begin{aligned}
&k_{2000} = \frac{1}{2i}(-s_{2000}^{(1)} - s_{0020}^{(1)} + s_{1010}^{(1)}), k_{0200} = \frac{1}{2i}(-s_{0200}^{(1)} - s_{0002}^{(1)} + s_{0101}^{(1)}),\\
&k_{0020} = \frac{1}{2i}(s_{2000}^{(1)} + s_{0020}^{(1)} + s_{1010}^{(1)}), k_{0002} = \frac{1}{2i}(s_{0200}^{(1)} + s_{0002}^{(1)} + s_{0101}^{(1)}),\\
&k_{1100} = \frac{1}{2i}(-s_{1100}^{(1)} + s_{1001}^{(1)} + s_{0110}^{(1)} + s_{0011}^{(1)}), k_{1010} = s_{2000}^{(1)} - s_{0020}^{(1)}\\
&k_{1001} = \frac{1}{2}(s_{1100}^{(1)} + s_{1001}^{(1)} - s_{0110}^{(1)} - s_{0011}^{(1)}), k_{0110} = \frac{1}{2}(s_{1100}^{(1)} - s_{1001}^{(1)} + s_{0110}^{(1)} - s_{0011}^{(1)}),\\
&k_{0101} = s_{0200}^{(1)} - s_{0002}^{(1)}, k_{0011} = \frac{1}{2i}(s_{1100}^{(1)} + s_{0110}^{(1)} + s_{1001}^{(1)} + s_{0011}^{(1)}),
\end{aligned} \tag{43}$$

Thus the normal form of the Hamiltonian H_2 isobtained as givenbyequation (38) correct to first order of eccentricity.

5. Study of Resonance Cases

In this section, we shall employ the KAM-theorem for stability and examine the existence of resonances of the third and fourth order. In order to study the resonances for different values of *e*, we shall need the value of λ_1 and λ_2 to $O(e^2)$, since$\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. The quantities $\lambda_1^{(2)}$ and$\lambda_2^{(2)}$ are found by the periodicity conditions of the functions $s_{1010}^{(2)}$ and $s_{0101}^{(2)}$. Equating the coefficients of e^2 in the expansion of equation (34) and integrating w.r.t f, we shall get:

$$s_{1010}^{(2)} = -2i \sin f (4\, a_{0020}^{''} s_{2000}^{(1)} + a_{1010}^{''} s_{1010}^{(1)} + a_{0011}^{''} s_{0011}^{(1)}) + \frac{i}{2} a_{1010}^{''} \sin 2f + i(a_{1010}^{''} + \lambda_1^{(2)}) f \tag{44}$$

$$\begin{aligned}
s_{0101}^{(2)} = \quad & -2i \sin f (4\, a_{0002}^{''} s_{0200}^{(1)} + a_{0011}^{''} s_{1100}^{(1)} + a_{0110}^{''} s_{1001}^{(1)} + a_{0101}^{''} s_{0101}^{(1)}) + \\
& \frac{i}{2} a_{0101}^{''} \sin 2f + i(a_{0101}^{''} + \lambda_2^{(2)}) f
\end{aligned}$$

Using the periodicity of$s_{1010}^{(2)}$ and $s_{0101}^{(2)}$and the equations (19), (22), (25), and (28), the values of$\lambda_1{}^{(2)}$and$\lambda_2{}^{(2)}$are obtained with the help of software Mathematica 10. Assuming the value of μ

giving the resonance $k_1\lambda_1 + k_2\lambda_2 = N$ be given as $\mu = \mu^{(0)} + e^2\mu^{(2)}$; taken correct to O($e^2$), where $\mu^{(0)}$ denotes the value of μ when the eccentricity is assumed to be zero and $\mu^{(2)}$; denotes the value of μ when the eccentricity $e \neq 0$: Also the value λ_1 and λ_1 are taken as function of μ. Then employing the Taylor's theorem expansion of λ_1 and λ_2 given as:

$$\lambda_1 = \lambda_1^{(0)} + e^2{\lambda_1}^{(2)} + e^2\mu^{(2)}\left(\frac{d\lambda_1}{d\mu}\right)_0$$

and

$$\lambda_2 = \lambda_2^{(0)} + e^2{\lambda_2}^{(2)} + e^2\mu^{(2)}\left(\frac{d\lambda_2}{d\mu}\right)_0 \tag{45}$$

where $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ denoting teh value of λ corresponding to $\mu = \mu^{(0)}$ can be obtained from (17). Substituting the value of λ_1 and λ_2 from equation (45) in $k_1\lambda_1 + k_2\lambda_2 = N$ and equating the coefficient of e^2 to zero, we get:

$$\mu^{(2)} = \left(\frac{k_1{\lambda_1}^{(2)} + k_2{\lambda_2}^{(2)}}{k_2\frac{dw_2}{d\mu} - k_1\frac{dw_1}{d\mu}}\right)_{\mu=\mu^{(0)}} \tag{46}$$

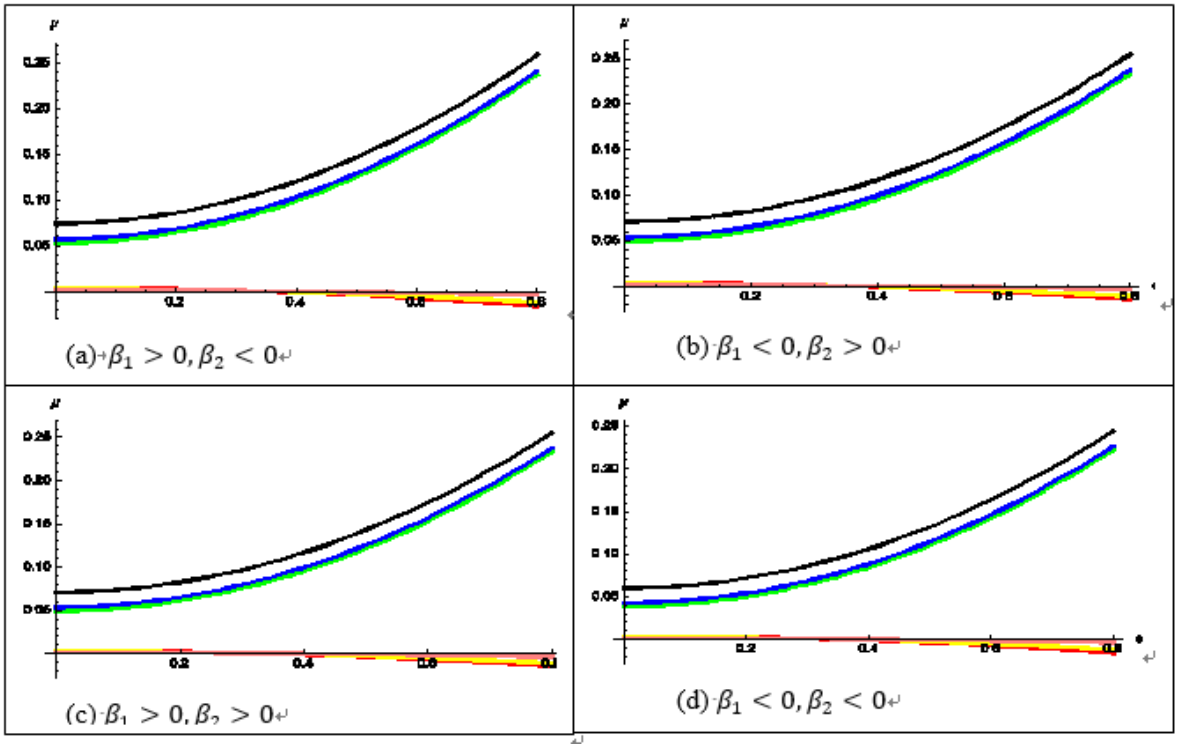

Fig.1. μ versus e for $3\lambda_2 = -1$

Taking our clue from Kumar and Choudhry [15], we have studied four cases of third order resonance graphically. Figures 1-4 show the value of μand $\mu^{(e)}$ as functions of e, taking the

values A_1=0.001, A_2=0.001, α=0.0005 and A_3 is varied as A_3=0, A_3= 0.01, A_3=0.05 for the cases of third order resonance:$3\lambda_2 = -1$, $3\lambda_2 = -2$, $\lambda_1+2\lambda_2 = 0$ and $\lambda_1 - 2\lambda_2 = 2$.

6. Discussion and Conclusion

The resonance cases and the linear stability of the elliptic restricted three body problema where both the primaries are luminous and all the three participating bodies are oblate has been analyzed. The Hamiltonian for the system is defined and then normalized using the Markeev's method [19].

Four particular cases of third order resonance are studied numerically and represented by figures 1-4. In figure 1, μ and $\mu^{(e)}$ are plotted w.r.t the eccentricity of the orbit of the primaries for the resonance case $3\lambda_2=-1$. It is observed that the value of μ is always less than the value of $\mu^{(e)}$ which is the required condition for stability, however for $e > 0.3$ the value of μ becomes negative when $\beta_1 > 0$, $\beta_2 < 0$; $\beta_1 < 0$, $\beta_2 > 0$ and $\beta_1 > 0$, $\beta_2 > 0$. For $\beta_1 < 0$, $\beta_2 < 0$ the value of μ becomes negative for $e > 0.35$. Similar pattern is observed in the case of $3\lambda_2 = -2$ and $\lambda_1 + 2\lambda_2 = 0$ as shown in Figure 2 and 3. But in the case $\lambda_1 - 2\lambda_2 = 2$, the value of μ is not obtained in the real $\mu - e$ plane. Thus, we conclude that the linear stability is observed in the resonance cases $3\lambda_2 = -1$, $3\lambda_2 = -2$ and $\lambda_1 + 2\lambda_2 = 0$, where as the triangular points are found to be linearly unstable in the case $\lambda_1 - 2\lambda_2 = 2$.

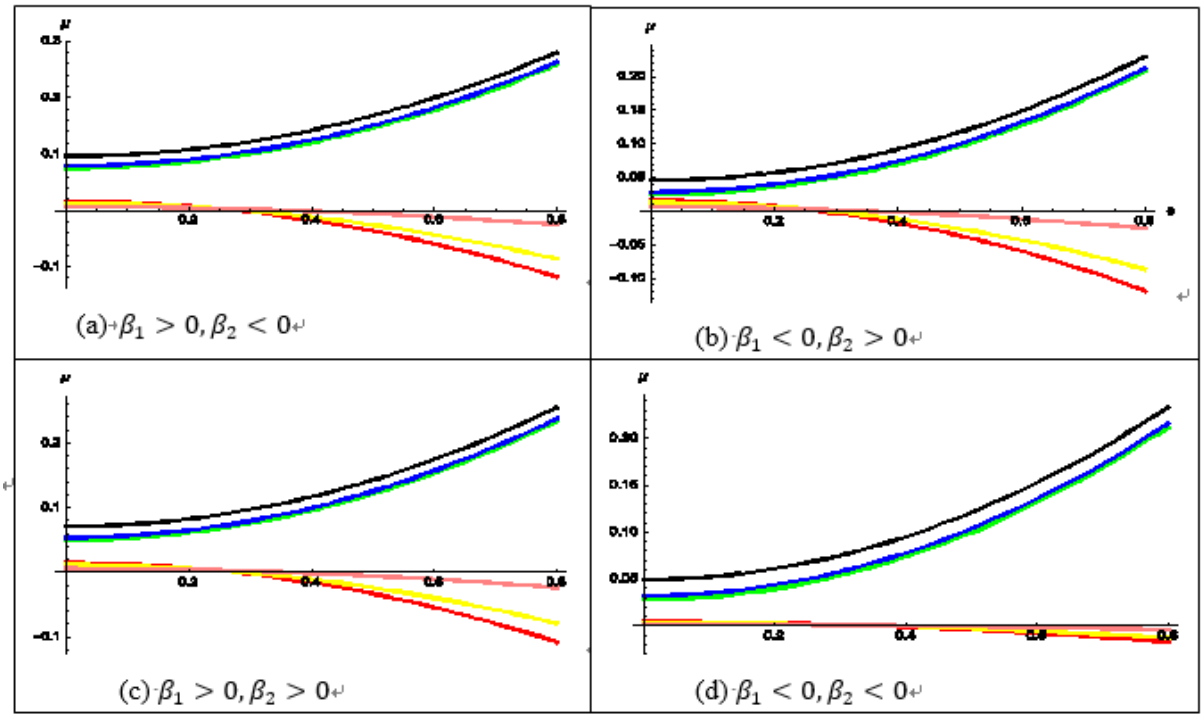

Fig.2. μ versus e for $3\lambda_2 = -2$

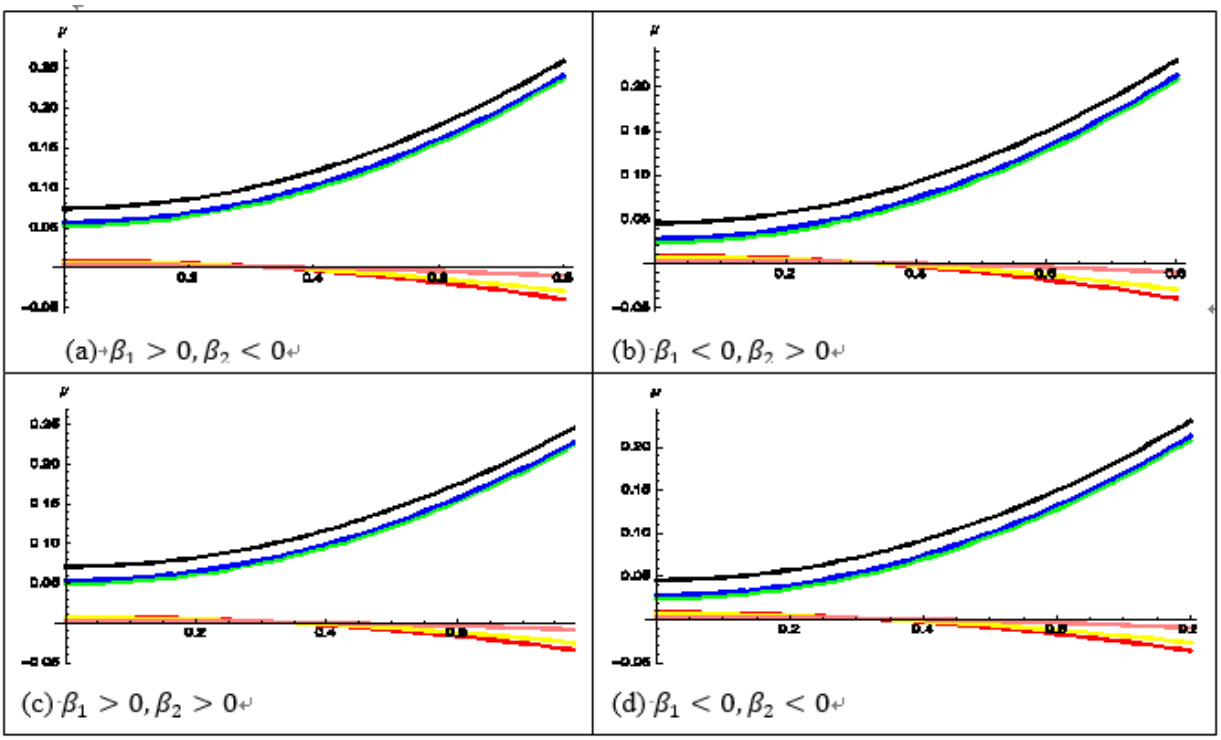

Fig. 3. μ versus e for $\lambda_1+2\lambda_2 =0$

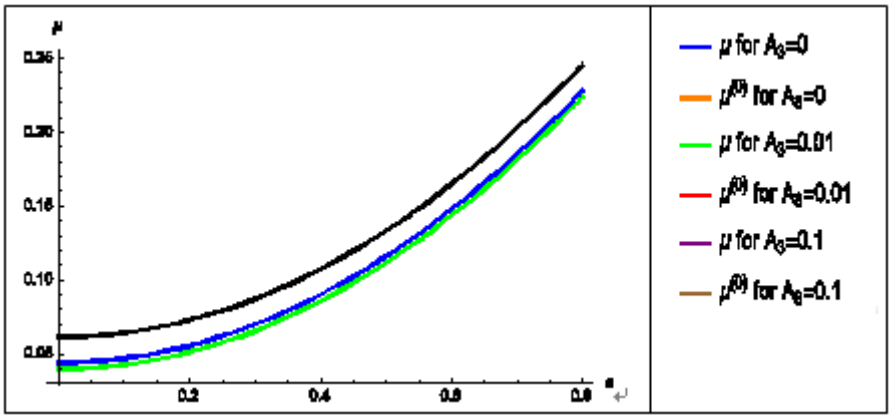

Fig. 4. μ versus e for $\lambda_1 -2\lambda_2 =2$

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