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Continuous Mappings and Fixed-Point Theorems in Probabilistic Normed Space

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Abstract

The notion of probabilistic normed space has been redefined by C. Alsina, B. Schweizer and A. Sklar [2]. But the results about the continuous operator in this space are not many. In this paper, we study B-contractions, H-contractions and strongly ε-continuous mappings and their respective relation to the strongly continuous mappings, and give some fixed-point theorems in this space.

Key words

Probabilistic Normed (PN) Space, Fixed-point theorem, Strongly ε-continuous.

1. Introduction

In 1963, Šerstnev [1] introduced Probabilistic Normed spaces, whose definition was generalized by C. Alsina, B. Schweizer and A. Sklar [2] in 1993. In this paper we adopt this generalized definition and the notations and concepts used are those of [2-6].

A distribution function (briefly, d.f.) is a function F from the extended real line $\overline{R} = [-\infty, +\infty]$ into the unit interval I=[0,1] that is left continuous nondecreasing and satisfies $F(-\infty) = 0$ and $F(\infty) = 1$. The set of all distribution functions will be denoted by Δ and the subset of those distribution functions called positive distribution functions such that F(0)=0, by Δ^+. By setting

$F \le G$ whenever $F(x) \le G(x)$ for all x in $\overline{R}$, a natural ordering in Δ and in Δ^+ has been introduced. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(x) = \begin{cases} 0, x \le 0 \\ 1, x > 0. \end{cases} \tag{1}$$

A triangle function is a binary operation on Δ^+, namely a function τ : $\Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative and nondecreasing, and which has ε_0 as a unit, that is, for all $F, G, H \in \Delta^+$, we have:

$$\tau(\tau(F,G),H) = \tau(F,\tau(G,H)), \tau(F,G) = \tau(G,F),$$
$$\tau(F,H) \le \tau(G,H), whenever F \le G, \tau(F,\varepsilon_0) = F.$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Typical continuous triangle functions are operations τ_T and τ_{T^*}, which are respectively given by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)), \tag{2}$$

and

$$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t)), \tag{3}$$

for all F, G in Δ^+ and all x in $\overline{R}$ [7, Sections7.2 and 7.3], and T is a continuous t-norm, i.e., a continuous binary operation on [0,1] which is associative, commutative, nondecreasing and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on [0,1] that is related to continuous t-norm through

$$T^*(x, y) = 1 - T(1-x, 1-y). \tag{4}$$

The most important t-norms are function *W, Prod* and *M* which are defined, respectively, by $W(a,b) = max\{a+b-1,0\}, Prod(a,b) = ab, M(a,b) = min\{a,b\}$.

Throughout this paper, we always assume that the t-norm *T* satisfies
$\sup_{t\in(0,1)} T(t,t) = 1$.

Definition 1.1.[7] A probabilistic metric (briefly, PM) space is a triple (S , F , τ), where S is a nonempty set, τ is a triangle function, and F is a mapping form $S\times S$ into Δ^+ such that, if F_{pq} denotes the value of F at the pair (p,q), the following conditions hold for all p,q and r in S:

(PM1) $F_{pq} = \varepsilon_0$ if and only if $p = q$; (θ is the null vector in S)

(PM2) $F_{pq} = F_{qp}$;

(PM2) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.

Definition 1.2.[2] A probabilistic normed space is a quadruple (V , υ , τ , τ^*), where *V* is a real vector space, τ and τ^* are continuous triangle functions and υ is a mapping from *V* into Δ^+ such that for all p, q in *V*, the following conditions hold:

(PN1) $\upsilon_p = \varepsilon_0$ if, and only if, $p = \theta$; (θ is the null vector in *V*)

(PN2) $\forall p \in V$, $\upsilon_{-p} = \upsilon_p$;

(PN3) $\upsilon_{p+q} \geq \tau(\upsilon_p, \upsilon_q)$;

(PN4) $\upsilon_p \leq \tau^*(\upsilon_{ap}, \upsilon_{(1-a)p})$ for all a in [0,1].

A Menger PN space under *T* is a PN space (V , υ , τ , τ^*), denoted by (*V*, *v*, *T*), in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm *T* and its t-conorm T^*.

The PN space is called a Serstnev space if the inequality (PN4) is replaced by the equality $\upsilon_p = \tau_M(\upsilon_{ap}, \upsilon_{(1-a)p})$, and, as a consequence, a condition stronger than (PN2) holds, namely $\upsilon_{\lambda p}(x) = \upsilon_p(\frac{x}{|\lambda|})$, for all $p \in V, \lambda \neq 0$ and $x \in R$, i.e., the (Š) condition (see [2]). The pair (V , υ) is said to be a Probabilistic Seminormed Space (briefly, PSN space) if $\upsilon : V \to \Delta^+$ satisfies (PN1) and (PN2).

Let { $p_n\}_{n=1}^{\infty}$ be a sequence of points in *V*. A is a sequence that converges to p in *V*, if for each $t > 0$, there is a positive integer *N* such that $\upsilon_{p_n - p}(t) > 1 - t$ for $n > N$, and is a Cauchy sequence, if for each $t > 0$ there is a positive integer N such that $\upsilon_{p_n - p_m}(t) > 1 - t$ for all $n, m > N$. A PN space is complete if every Cauchy sequence converges.

Definition 1.3.[7] A PSN space (V, υ) is said to be equilateral if there is a d.f. $F \in \Delta^+$ different from ε_0 and from $\varepsilon_{+\infty}$, such that, for every $p \neq \theta$, $\upsilon_p = F$. Therefore, every equilateral PSN space (V, υ) is a PN space under $\tau = \tau* = \tau_M$, where the triangle function is defined for $G, H \in \Delta^+$ by

$$\tau_M(G, H)(x) = \sup_{s+t=x} min\{G(s), H(t)\}.$$

An equilateral PN space will be denoted by (V, F, M).

Definition 1.4.[8] Let $(V, \upsilon, \tau, \tau^*)$ be a PN space, for $p \in V$ and $\lambda \in (0,1)$. We give the following two conditions:

(Z_1) For all $a \in (0,1)$, there exists a $\beta \in [1, \infty[$ such that

$$\upsilon_p(\lambda) > 1 - \lambda \textit{ implies } \upsilon_{ap}(a\lambda) > 1 - \frac{a}{\beta}\lambda.$$

(Z_2) For all $a \in (0,1)$, let $\beta_0(a, \lambda) = \frac{1+\sqrt{1-4a(1-a)\lambda}}{2}$, then

$$\upsilon_p(\lambda) > 1 - \lambda \textit{ implies } \upsilon_{ap}(a\lambda) > 1 - \frac{a}{\beta_0(a, \lambda)}\lambda.$$

Definition 1.5.[7] There is a natural topology in the PN space $(V, \upsilon, \tau, \tau^*)$, and it is called strongly topology, defined by the following neighborhoods: $N_p(\lambda) = \{q \in V : \upsilon_{q-p}(\lambda) > 1 - \lambda\}$,

where λ>0. The strongly neighborhood system for V is the union $\cup_{p \in V} N_p$, where $N_p = \{N_p(\lambda); \lambda > 0\}$. In the strongly topology, the closure $\overline{N_p(\lambda)}$ of $N_p(\lambda)$ is defined by

$\overline{N_p(\lambda)} := N_p(\lambda) \bigcup N_p(\lambda)$, where $N_p(\lambda)$ is the set of limit points of all convergent sequences in $N_p(\lambda)$. From [5, Theorem 3], we know every PN space (V, v, τ, τ^*) has a completion. C.Alsina, B.Schweizer and A. Sklar [3, Theorem 1] have proved that υ is a uniformly continuous mapping from V into Δ^+.

Now, we give two different definitions of the contractions in PN space.

Definition 1.6.[7](i).A mapping $f:(V,\upsilon,\tau,\tau^*)\to(U,\mu,\sigma,\sigma^*)$ is a B-contraction, if there is a constant $k\in(0,1)$ such that for all p and q in *V*, and all x>0,

$$\mu_{f(p)-f(q)}(kx)\geq\upsilon_{p-q}(x). \tag{5}$$

(ii). A mapping $f:(V,\upsilon,\tau,\tau^*)\to(U,\mu,\sigma,\sigma^*)$ is an H-contraction, if there is a constant $k\in(0,1)$ such that for p and q in *V*, and all x>0,

$$\upsilon_{p-q}(x)>1-x \textit{ implies } \mu_{f(p)-f(q)}(kx)>1-kx. \tag{6}$$

Remark 1.1. If f is a linear operator, for all $p\in V$, we have that (1.5) is equivalent to $\mu_{f(p)}(kx)\geq\upsilon_p(x)$ and (1.6) is equivalent to that

$\upsilon_p(x)>1-x \textit{ implies } \mu_{f(p)}(kx)>1-kx.$

Definition 1.7. [6] Given a nonempty set A in a PN space (V,υ,τ,τ^*), the probabilistic radius R_A of A is defined by

$$R_A(x):=\begin{cases}\ell^-\varphi_A(x), x\in[0,+\infty[,\\ 1, x=+\infty,\end{cases} \tag{7}$$

where $\ell^- f(x)$ denotes the left limit of the function *f* at the point x and

$\varphi_A(x):=\inf\{\nu_p(x):p\in A\}.$

As a consequence of DEFINITION 1.7., we have $\upsilon_p\geq R_A$ for all $p\in A$.

Definition 1.8. [9] In a PN space (V,υ,τ,τ^*), a mapping $f:V\to V$ is said to be strongly ε-continuous $(\varepsilon>0)$, if for each $p\in V$, it admits a strong λ-neighborhood $N_p(\lambda)$ such that

$R_{f(N_p(\lambda))}(\varepsilon)>1-\varepsilon.$

Lemma 1.9. [9] Suppose (V,υ,τ,τ^*) be a PN space and $A\subset V$. If $f:A\to A$ is strongly ε-continuous, then for each $p\in A$ and $\varepsilon>0$, we have

$\upsilon_{f(p)}(\varepsilon)>1-\varepsilon.$

2. Main Results

Definition 2.1. A mapping $f:(V,\upsilon,\tau,\tau^*)\to(U,\mu,\sigma,\sigma^*)$ is strongly continuous, if for any $\varepsilon>0$, there exists $\delta>0$ such that

$$q\in N_p(\delta)\Rightarrow f(q)\in N_{f(p)}(\varepsilon), \tag{8}$$

where *(V, ν, τ, τ*)* and (U,μ,σ,σ^*) are PN spaces, and $p,q\in V\setminus\{\theta\}$.

Theorem 2.1. In a PN space *(V, ν, τ, τ*)* with $\tau\geq\tau_W$, a strongly ε-continuous mapping $f:V\to V$ is strongly continuous.

Proof. Let $\varepsilon<1/2$. In view of Definition 1.8, there exists $\delta>0$ such that $R_{f(N_p(\delta))}(\varepsilon/2)>1-\varepsilon/2$,

therefore $q\in N_p(\delta)\Rightarrow\upsilon_{f(q)}(\varepsilon/2)\geq R_{f(N_p(\delta))}(\varepsilon/2)>1-\varepsilon/2$, i.e.,

$\upsilon_{p-q}(\delta)>1-\delta$ *implies* $\upsilon_{f(q)}(\varepsilon/2)>1-\varepsilon/2$. From $p\in N_p(\delta)$, we have $\upsilon_{f(p)}(\varepsilon/2)\geq R_{f(N_p(\delta))}(\varepsilon/2)>1-\varepsilon/2$, thus

$$\begin{aligned}\upsilon_{f(p)-f(q)}(\varepsilon)&\geq\tau(\upsilon_{f(p)},\upsilon_{f(q)})(\varepsilon)\\&\geq\tau_W(\upsilon_{f(p)},\upsilon_{f(q)})(\varepsilon)\\&=\sup_{s+t=\varepsilon}W(\upsilon_{f(p)}(s),\upsilon_{f(q)}(t))\\&\geq W(\upsilon_{f(p)}(\varepsilon/2),\upsilon_{f(q)}(\varepsilon/2))\\&\geq W(1-\varepsilon/2,1-\varepsilon/2)\\&=1-\varepsilon\end{aligned}$$

i.e., $f(q)\in N_{f(p)}(\varepsilon)$. So $\forall q\in N_p(\delta)\Rightarrow f(q)\in N_{f(p)}(\varepsilon)$.

Theorem 2.2. Let *(V, ν, τ, τ*)* be a PN space, then

(i). A B-contraction mapping is strongly continuous;

(ii). an H-contraction mapping is strongly continuous.

Proof. (i). Suppose *(V, ν, τ, τ*)* be a PN space and $f:V\to V$ be B-contraction. According to Definition 1.6, there is a constant $k\in(0,1)$ such that for p and q in *V*, and x>0

$$\upsilon_{f(p)-f(q)}(kx)\geq\upsilon_{p-q}(x). \tag{9}$$

Therefore, let a>1, we have

$$\upsilon_{f(p)-f(q)}(ax) \geq \upsilon_{f(p)-f(q)}(kx) \geq \upsilon_{p-q}(x). \tag{10}$$

Let $v_{p-q}(x)>1-x$ we have

$$\upsilon_{f(p)-f(q)}(ax) \geq \upsilon_{p-q}(x) > 1-x > 1-ax, \tag{11}$$

i.e.,

$$q \in N_p(x) \Rightarrow f(q) \in N_{f(p)}(ax). \tag{12}$$

So for ε>0, set $\delta = \varepsilon / a$ such that

$$q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon). \tag{13}$$

By Definition 2.1., we have that f is strongly continuous.

(ii). Suppose *(V, ν, τ, τ*)* be a PN space and $f: V \to V$ be H-contraction, and if ε>0, in view of Definition 1.6, there is a constant $k_0 \in (0,1)$ such that for p and q in *V*,

$$\upsilon_{p-q}(\varepsilon / k_0) > 1-\varepsilon / k_0 \text{ implies } \upsilon_{f(p)-f(q)}(\varepsilon) > 1-\varepsilon, \tag{14}$$

i.e.,

$$q \in N_p(\varepsilon / k_0) \Rightarrow f(q) \in N_{f(p)}(\varepsilon). \tag{15}$$

So for ε>0, set $\delta = \varepsilon / k_0$ such that

$$q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon). \tag{16}$$

Basing on Definition 2.1., we have proven that f is strongly continuous. □

The following examples, Example 2.1. and 2.2., show that a B-contraction isn't necessarily an H-contraction, an H-contraction isn't necessarily a B-contraction, and a strongly continues mapping isn't necessarily a B-contraction or an H-contraction.

Example 2.1. Let V be a vector space and $\upsilon_\theta = \mu_\theta = \varepsilon_0$, if $a \in (2,3)$, p, $q \in V$ (p, $q \neq \theta$) and $x \in \overline{R}$,

$$\upsilon_p(x) = \begin{cases} 0, x \leq a \\ 1, x > a \end{cases} \quad \mu_p(x) = \begin{cases} 0, x \leq 0 \\ 1/a, 0 < x \leq \frac{2a}{3} \\ 2/a, \frac{2a}{3} < x < \infty \\ 1, x = \infty \end{cases}$$

and if $\tau(\upsilon_p, \upsilon_q)(x) = \tau^*(\upsilon_p, \upsilon_q)(x) = \sup_{s+t=x} \min(\upsilon_p(s), \upsilon_q(t))$, then *(V, ν, τ, τ*)* and *(V, μ, τ, τ*)* are equilateral PN spaces by Definition 1.3. Now let I: *(V, ν, τ, τ*)* →*(V, μ, τ, τ*)* be the identity operator, then I is not a B-contraction, but an H-contraction. In fact, for every $k \in (0,1)$, $x>a$ and $p\neq\theta$, $\mu_{Ip}(kx) \leq \mu_{Ip}(x) = \mu_p(x) = \frac{2}{a} < 1 = \nu_p(x)$. Hence I is not a B-contraction.

Next we'll prove that I is an H-contraction. Suppose $\nu_p(x) > 1 - x$, where $p \neq \theta$. This condition holds only if $x > 1$. In fact, if $x \leq 1$, then $\nu_p(x) = 0 \leq 1 - x$. For $a \in (2,3)$, if $1 < x \leq a$, let $h = \frac{2}{3}$, then $\frac{2}{3} < hx \leq \frac{2a}{3}$, therefore $\mu_{Ip}(hx) = \mu_p(hx) = \frac{1}{a} > \frac{1}{3} = 1 - \frac{2}{3} > 1 - hx.$ If $x > a, let\ \ h = \frac{2}{3}$, then $hx > \frac{2a}{3}$, therefore $\mu_{Ip}(hx) = \mu_p(hx) = \frac{2}{a} > 1 - \frac{a}{2} > 1 - \frac{2a}{3} > 1 - hx.$ Thus there is a constant $h = \frac{2}{3}$ such that for all points $p \neq \theta$ in V ,and all $x > 0$,

$$\upsilon_p(x) > 1 - x \textit{ implies } \mu_{Ip}(hx) > 1 - hx, \tag{17}$$

i.e., I is an H-contraction. In view of Theorem 2.2. (ii), we have that I is strongly continuous.

Example 2.2. Let $V = V' = \overline{R}$, $\upsilon_0 = \mu_0 = \varepsilon_0$, if, for x>0, $p \neq 0$ and $a = \frac{k+3}{2}$, where $k \in (0,1)$,

$$\upsilon_p(x)=\begin{cases}0, x\le 0\\ \frac{1}{a}, 0<x\le a\\ 1, a<x\le\infty\end{cases}\quad \mu_p(x)=\begin{cases}0, x\le 0\\ \frac{1}{a}, 0<x\le \frac{a}{2}\\ 1, \frac{a}{2}<x\le\infty\end{cases}$$

and if $\tau(\upsilon_p,\upsilon_q)(x)=\tau^*(\upsilon_p,\upsilon_q)(x)=\sup_{s+t=x}\min(\upsilon_p(s),\upsilon_q(t)))$, then $(\overline{R},\upsilon,\tau,\tau^*)$ and $(\overline{R},\mu,\tau,\tau^*)$ are equilateral PN spaces by Definition 1.3. Now let I: $(\overline{R},\upsilon,\tau,\tau^*)\to(\overline{R},\mu,\tau,\tau^*)$ be the identity operator, then I is not an H-contraction, but a B-contraction. In fact, for every $k\in(0,1)$, we have that $a=\frac{k+3}{2}\in(\frac{3}{2},2)$. Let $x=\frac{1}{a}$, we have that $\upsilon_p(x)=\upsilon_p(\frac{1}{a})=\frac{1}{a}>1-\frac{1}{a}=1-x$. But,

$$\mu_{Ip}(kx)\le\mu_{Ip}(x)=\mu_{Ip}(\frac{1}{a})=\mu_p(\frac{1}{a})=\frac{1}{a}<1-\frac{k}{a}=1-kx.$$

Hence I is not an H-contraction. Meanwhile, for every $p\in\overline{R}$ and x>0, there exists a constant $k_0=\frac{2}{3}$ such that

$$\mu_{Ip}(k_0x)=\mu_{Ip}(\frac{2x}{3})=\mu_p(\frac{2x}{3})\ge\mu_p(\frac{x}{2})=\begin{cases}0, x\le 0\\ \frac{1}{a}, 0<x\le a\\ 1, a<x\le\infty\end{cases}=\upsilon_p(x),$$

i.e., I is a B-contraction. In view of Theorem 2.2.(ii), I is strongly continuous.

Example 2.3. Let PN space *(V, v, τ, τ*)* and (V,μ,τ,τ^*) satisfy Example 2.1, and I: $(V,\upsilon,\tau,\tau^*)\to(V,\mu,\tau,\tau^*)$ be the identity operator, then I is not strongly ε-continuous, but strongly continuous. In fact, according to Example 2.1., it is obvious that I is strongly continuous.

Now we are going to prove that I is not strongly ε-continuous. Suppose I is strongly ε-continuous. Let $A\subset V$ be not empty. In view of Lemma 1.1., for each $p\in A$ and $\varepsilon>0$, we have $\mu_{Ip}(\varepsilon)>1-\varepsilon$. However, let $\varepsilon_0\in(0,\frac{1}{3})$, for each $p\in A$ and $p\ne 0$, we have

$\mu_{Ip}(\varepsilon_0) = \mu_p(\varepsilon_0) \le \mu_p(\frac{1}{3}) = \frac{1}{a} < \frac{2}{3} < 1 - \varepsilon_0$. Thus, there appears a contradiction. So, we have that I is not strongly ε-continuous.

Lemma 2.1. [10] Let *V* be Banach space and *D* be a compact and convex subset of *V*. If $f: D \to D$ is a strongly continuous mapping, then f has at least one fixed point on *D*.

Not all PN spaces are Banach spaces; Lemma 2.2. shows that under some conditions, a PN space is a Banach space.

Lemma 2.2. [8] Let *(V, ν, τ, τ*)* be a TV PN space and $N_\theta(\lambda)$ be strong λ-neighborhoods of θ, where $\lambda \in (0,1)$.

(i) Suppose $\tau \ge \tau_W$. If there is an $N_\theta(\lambda)$ satisfying (Z_1), then $(V, \upsilon, \tau, \tau^*)$ is nomable.

(ii) Suppose $\tau \ge \tau_\pi$, $(\pi = Prod)$. If there is an $N_\theta(\lambda)$ satisfying (Z_2), then $(V, \upsilon, \tau, \tau^*)$ is nomable.

Theorem 2.3. Let A be a compact and convex subset of TV PN space $(V, \upsilon, \tau, \tau^*)$ and $f: A \to A$ be a strongly continuous mapping.

(i) Suppose $\tau \ge \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_1), then f has at least one fixed point on A.

(ii) Suppose $\tau \ge \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_2), then f has at least one fixed point on A.

Proof. In view of Lemma 2.1. and Lemma 2.2., it is obvious that Theorem 2.3. holds.

Corollary 2.1. Let A be a compact and convex subset of TV PN space *(V, ν, τ, τ*)* and $f: A \to A$ be a B-contraction or an H-contraction mapping.

(i) Suppose $\tau \ge \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_1), then *f* has at least one fixed point on *A*.

(ii) Suppose $\tau \ge \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_2), then *f* has at least one fixed point on *A*.

Proof. In view of Theorem 2.2., we have that $f: A \to A$ is a strongly continuous mapping on A. By Theorem 2.3., *f* has at least one fixed point on *A*.

Corollary 2.2. Let *A* be a compact and convex subset of TV PN space *(V, ν, τ, τ*)* and $f: A \to A$ be a strongly ε-continuous mapping.

(i) Suppose $\tau \geq \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_1), then *f* has at least one fixed point on A.

(ii) Suppose $\tau \geq \tau_W$ and there is an $N_\theta(\lambda)$ satisfying (Z_1), then *f* has at least one fixed point on A.

Proof. In view of Theorem 2.1., we have that $f: A \to A$ is a strongly continuous mapping on A. By Theorem 2.3., we have that *f* has at least one fixed point on A.

Theorem 2.4. Let A be a compact and convex subset of PN space *(V, ν, τ, τ*)*, where *(V, ν, τ, τ*)* is a Banach space. If $f: A \to A$ is a strongly continuous mapping, then f has at least one fixed point on A.

Proof. In view of Lemma 2.1., it is obvious that Theorem 2.4. holds. □

Let *(V, ν, τ, τ*)* be a PN space and $f: V \to V$ be a single-valued self mapping. A point $p \in V$ with the property $\upsilon_{f(p)-p} = \varepsilon_0$ is called a fixed point of f on V. Note that, for every $p \in V / \{\theta\}$, if $\upsilon_{f(p)-p}(t) < 1$ for all $t > 0$ (see [12], Example 2.4.), then $f(p) \neq p$, i.e., f has no fixed point on V. In such a situation a question arises about the existence of an approximate fixed point. The following is the definition of the approximate fixed point in PN space.

Definition 2.2. [9] Suppose *(V, ν, τ, τ*)*be a PN space and $A \subset V$. We call $p \in A$ an ε-fixed point of $f: A \to A$, if, there exists an $\varepsilon > 0$ such that $\sup_{t<\varepsilon} \upsilon_{f(p)-p}(t) = 1$. A self mapping $f: A \to A$ has approximate fixed point property (in short a.f.p.p.) if the function *f* possesses at least one ε-fixed point.

Definition 2.3. *A* is bounded, if for every $n \in N$ and for every $p \in A$, there is a $k \in N$ such that $\nu_{p/k}(1/n) > 1 - 1/n$.

Lemma 2.3. [3] If $|\alpha| \leq |\beta|$, then $\upsilon_{ap} \geq \upsilon_{\beta p}$.

Theorem 2.5. Suppose *A* be a bounded and convex subset of PN space *(V, ν, τ, τ*)* with $\tau \geq \tau_W$, where*(V, ν, τ, τ*)* is a Banach space. If the mapping $f: A \to A$ is strongly ε-continuous, then *f* has at least one approximate fixed-point on *A*.

Proof. Since *f* is an ε-continuous on *A*, by Definition 1.8. and Lemma 1.1, we have that for every $p \in A$, $\sup_{\varepsilon>0} \upsilon_{f(p)}(\varepsilon) = 1$. Let B be a compact and convex subset of *A*, defined by $B = (1-a)\overline{A}$, where $\overline{A}$ is a closure of *A* and (0<a<1) In view of Theorem 2.1., we have that *f* is strongly continuous. We can define a strongly continuous function $g: B \to B$ by

$g(p)=(1-a)f(p), \forall p \in B.$ By Theorem 2.4., there is a $p_0 \in B$ such that $g(p_0)=p_0$, which implies $(1-a)f(p_0)=p_0$. Whence $\upsilon_{(1-a)f(p_0)-p_0}=\varepsilon_0$. Since $f(p_0)-p_0=(1-a)f(p_0)-p_0+af(p_0)$, by (PN3) and Lemma 2.3., we have

$$\begin{aligned}\upsilon_{f(p_0)-p_0} &\geq \tau(\upsilon_{(1-a)f(p_0)-p_0}, \upsilon_{af(p_0)}) \\ &= \tau(\varepsilon_0, \upsilon_{f(p_0)}) \\ &= \upsilon_{f(p_0)}.\end{aligned}$$

By taking sup over *0<t<ε* on both sides of the inequality, we have $\sup_{0<t<\varepsilon}\upsilon_{f(p_0)-p_0}(t) \geq \sup_{0<t<\varepsilon}\upsilon_{f(p_0)}(t)$.

Because $p_0 \in B \subset A$, $\sup_{0<t<\varepsilon}\upsilon_{f(p_0)}(t)=1$. So $\sup_{0<t<\varepsilon}\upsilon_{f(p_0)-p_0}(t) \geq \sup_{0<t<\varepsilon}\upsilon_{f(p_0)}(t)=1$. According to **Definition 2.2.** p_0 is an approximate fixed point of *f*, thus *f* has at least one ε-fixed-point on *A*.□

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