Continuous Mappings and Fixed-Point Theorems in Probabilistic Normed Space

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Abstract

The notion of probabilistic normed space has been redefined by C. Alsina, B. Schweizer and A. Sklar [2]. But the results about the continuous operator in this space are not many. In this paper, we study B-contractions, H-contractions and strongly $\varepsilon$-continuous mappings and their respective relation to the strongly continuous mappings, and give some fixed-point theorems in this space.

Key words

Probabilistic Normed (PN) Space, Fixed-point theorem, Strongly $\varepsilon$-continuous.

1. Introduction

In 1963, Šerstnev [1] introduced Probabilistic Normed spaces, whose definition was generalized by C. Alsina, B. Schweizer and A. Sklar [2] in 1993. In this paper we adopt this generalized definition and the notations and concepts used are those of [2-6].

A distribution function (briefly, d.f.) is a function $F$ from the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I=[0,1]$ that is left continuous nondecreasing and satisfies $F(-\infty) = 0$ and $F(\infty) = 1$. The set of all distribution functions will be denoted by $\Delta$ and the subset of those distribution functions called positive distribution functions such that $F(0)=0$, by $\Delta^+$. By setting
\( F \leq G \) whenever \( F(x) \leq G(x) \) for all \( x \) in \( \overline{R} \), a natural ordering in \( \Delta \) and in \( \Delta^+ \) has been introduced. The maximal element for \( \Delta^+ \) in this order is the distribution function given by

\[
\varepsilon_0(x) = \begin{cases} 
0, & x \leq 0 \\
1, & x > 0.
\end{cases}
\]  

(1)

A triangle function is a binary operation on \( \Delta^+ \), namely a function \( \tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+ \) that is associative, commutative and nondecreasing, and which has \( \varepsilon_0 \) as a unit, that is, for all \( F, G, H \in \Delta^+ \), we have:

\[
\tau(\tau(F,G),H) = \tau(F,\tau(G,H)), \quad \tau(F,G) = \tau(G,F),
\]

\[
\tau(F,H) \leq \tau(G,H), \text{ whenever } F \leq G, \quad \tau(F,\varepsilon_0) = F.
\]

Continuity of a triangle function means continuity with respect to the topology of weak convergence in \( \Delta^+ \).

Typical continuous triangle functions are operations \( \tau_T \) and \( \tau_{T^*} \), which are respectively given by

\[
\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),
\]

(2)

and

\[
\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t)),
\]

(3)

for all \( F, G \) in \( \Delta^+ \) and all \( x \) in \( \overline{R} \) \[7, \text{Sections} \text{7.2 and 7.3}\], and \( T \) is a continuous t-norm, i.e., a continuous binary operation on \([0,1]\) which is associative, commutative, nondecreasing and has 1 as identity; \( T^* \) is a continuous t-conorm, namely a continuous binary operation on \([0,1]\) that is related to continuous t-norm through

\[
T^*(x, y) = 1 - T(1 - x, 1 - y).
\]

(4)
The most important t-norms are function $W$, Prod and $M$ which are defined, respectively, by $W(a, b) = \max\{a + b - 1, 0\}$, $\text{Prod}(a, b) = ab$, $M(a, b) = \min\{a, b\}$.

Throughout this paper, we always assume that the t-norm $T$ satisfies $\sup_{t \in (0, 1)} T(t, t) = 1$.

**Definition 1.1.** [7] A probabilistic metric (briefly, PM) space is a triple $(S, F, \tau)$, where $S$ is a nonempty set, $\tau$ is a triangle function, and $F$ is a mapping from $S \times S$ into $\Delta^+$ such that, if $F_{pq}$ denotes the value of $F$ at the pair $(p, q)$, the following conditions hold for all $p, q$ and $r$ in $S$:

- (PM1) $F_{pq} = \varepsilon_0$ if and only if $p = q$; ($\theta$ is the null vector in $S$)
- (PM2) $F_{pq} = F_{qp}$;
- (PM2) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.

**Definition 1.2.** [2] A probabilistic normed space is a quadruple $(V, \nu, \tau, \tau^*)$, where $V$ is a real vector space, $\tau$ and $\tau^*$ are continuous triangle functions and $\nu$ is a mapping from $V$ into $\Delta^+$ such that for all $p, q$ in $V$, the following conditions hold:

- (PN1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$; ($\theta$ is the null vector in $V$)
- (PN2) $\forall p \in V$, $\nu_{-p} = \nu_p$;
- (PN3) $\nu_{pq} \geq \tau(\nu_p, \nu_q)$;
- (PN4) $\nu_p \leq \tau^*(\nu_{ap}, \nu_{(1-a)p})$ for all $a$ in $[0, 1]$.

A Menger PN space under $T$ is a PN space $(V, \nu, \tau, \tau^*)$, denoted by $(V, \nu, T)$, in which $\tau = \tau_T$ and $\tau^* = \tau^*_T$, for some continuous t-norm $T$ and its t-conorm $T^*$.

The PN space is called a Serstnev space if the inequality (PN4) is replaced by the equality $\nu_p = \tau_M(\nu_{ap}, \nu_{(1-a)p})$, and, as a consequence, a condition stronger than (PN2) holds, namely $\nu_{ap}(x) = \nu_p(\lambda x)$, for all $p \in V, \lambda \neq 0$ and $x \in R$, i.e., the (Š) condition (see [2]). The pair $(V, \nu)$ is said to be a Probabilistic Seminormed Space (briefly, PSN space) if $\nu : V \rightarrow \Delta^+$ satisfies (PN1) and (PN2).

Let $\{p_n\}_{n=1}^\infty$ be a sequence of points in $V$. A is a sequence that converges to $p$ in $V$, if for each $t > 0$, there is a positive integer $N$ such that $\nu_{p_n - p}(t) > 1 - t$ for $n > N$, and is a Cauchy sequence,
if for each \( t > 0 \) there is a positive integer \( N \) such that \( v_{p_n - p_m}(t) > 1 - t \) for all \( n, m > N \). A PN space is complete if every Cauchy sequence converges.

**Definition 1.3.**[7] A PSN space \((V, \nu)\) is said to be equilateral if there is a d.f. \( F \in \Delta^+ \) different from \( \varepsilon_0 \) and from \( \varepsilon_{+\infty} \), such that, for every \( p \neq \theta \), \( \nu_p = F \). Therefore, every equilateral PSN space \((V, \nu)\) is a PN space under \( \tau = \tau^* = \tau_M \), where the triangle function is defined for \( G, H \in \Delta^+ \) by

\[
\tau_M(G, H)(x) = \sup_{s+t=x} \min\{G(s), H(t)}.
\]

An equilateral PN space will be denoted by \((V, F, M)\).

**Definition 1.4.**[8] Let \((V, \nu, \tau, \tau^*)\) be a PN space, for \( p \in V \) and \( \lambda \in (0, 1) \). We give the following two conditions:

\((Z_1)\) For all \( a \in (0, 1) \), there exists a \( \beta \in [1, \infty[ \) such that

\[
\nu_p(\lambda) > 1 - \lambda \implies \nu_{ap}(a\lambda) > 1 - \frac{a}{\beta} \lambda.
\]

\((Z_2)\) For all \( a \in (0, 1) \), let \( \beta(a, \lambda) = \frac{1 + \sqrt{4a(1-a)}}{2} \), then

\[
\nu_p(\lambda) > 1 - \lambda \implies \nu_{ap}(a\lambda) > 1 - \frac{a}{\beta(a, \lambda)} \lambda.
\]

**Definition 1.5.**[7] There is a natural topology in the PN space \((V, \nu, \tau, \tau^*)\), and it is called strongly topology, defined by the following neighborhoods: \( N_p(\lambda) = \{ q \in V : \nu_{q-p}(\lambda) > 1 - \lambda \} \), where \( \lambda > 0 \). The strongly neighborhood system for \( V \) is the union \( \bigcup_{p \in V} N_p \), where \( N_p = \{ N_p(\lambda) ; \lambda > 0 \} \). In the strongly topology, the closure \( \overline{N_p(\lambda)} \) of \( N_p(\lambda) \) is defined by

\[
\overline{N_p(\lambda)} := N_p(\lambda) \cup N_p(\lambda), \text{ where } N_p(\lambda) \text{ is the set of limit points of all convergent sequences in } N_p(\lambda).
\]

From [5, Theorem 3], we know every PN space \((V, \nu, \tau, \tau^*)\) has a completion. C. Alsina, B. Schweizer and A. Sklar [3, Theorem 1] have proved that \( \nu \) is a uniformly continuous mapping from \( V \) into \( \Delta^+ \).

Now, we give two different definitions of the contractions in PN space.
Definition 1.6.[7](i) A mapping \( f : (V, \nu, \tau, \tau^*) \rightarrow (U, \mu, \sigma, \sigma^*) \) is a B-contraction, if there is a constant \( k \in (0,1) \) such that for all \( p \) and \( q \) in \( V \), and all \( x > 0 \),

\[
\mu_{f(p)-f(q)}(kx) \geq \nu_{p-q}(x). \tag{5}
\]

(ii) A mapping \( f : (V, \nu, \tau, \tau^*) \rightarrow (U, \mu, \sigma, \sigma^*) \) is an H-contraction, if there is a constant \( k \in (0,1) \) such that for \( p \) and \( q \) in \( V \), and all \( x > 0 \),

\[
\nu_{p-q}(x) > 1 - x \implies \mu_{f(p)-f(q)}(kx) > 1 - kx. \tag{6}
\]

Remark 1.1. If \( f \) is a linear operator, for all \( p \in V \), we have that (1.5) is equivalent to

\[
\mu_{f(p)}(kx) \geq \nu_{p}(x) \text{ and (1.6) is equivalent to that}
\]

\[
\nu_{p}(x) > 1 - x \implies \mu_{f(p)}(kx) > 1 - kx.
\]

Definition 1.7. [6] Given a nonempty set \( A \) in a PN space \( (V, \nu, \tau, \tau^*) \), the probabilistic radius \( R_A \) of \( A \) is defined by

\[
R_A(x) := \begin{cases} 
\ell^- \varphi_A(x), x \in [0, +\infty[, \\
1, x = +\infty,
\end{cases}
\tag{7}
\]

where \( \ell^- f(x) \) denotes the left limit of the function \( f \) at the point \( x \) and

\[
\varphi_A(x) := \inf \{ \nu_p(x) : p \in A \}.
\]

As a consequence of DEFINITION 1.7., we have \( \nu_p \geq R_A \) for all \( p \in A \).

Definition 1.8. [9] In a PN space \( (V, \nu, \tau, \tau^*) \), a mapping \( f : V \rightarrow V \) is said to be strongly \( \varepsilon \)-continuous \( (\varepsilon > 0) \), if for each \( p \in V \), it admits a strong \( \lambda \)-neighborhood \( N_p(\lambda) \) such that

\[
R_{f(N_p(\lambda)}(\varepsilon) > 1 - \varepsilon.
\]

Lemma 1.9. [9] Suppose \( (V, \nu, \tau, \tau^*) \) be a PN space and \( A \subset V \). If \( f : A \rightarrow A \) is strongly \( \varepsilon \)-continuous, then for each \( p \in A \) and \( \varepsilon > 0 \), we have

\[
\nu_{f(p)}(\varepsilon) > 1 - \varepsilon.
\]
2. Main Results

Definition 2.1. A mapping \( f : (V, \nu, \tau, \tau^*) \to (U, \mu, \sigma, \sigma^*) \) is strongly continuous, if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon),
\]

where \((V, \nu, \tau, \tau^*)\) and \((U, \mu, \sigma, \sigma^*)\) are PN spaces, and \( p, q \in V \setminus \{\theta\} \).

Theorem 2.1. In a PN space \((V, \nu, \tau, \tau^*)\) with \( \tau \geq \tau_w \), a strongly \( \varepsilon \)-continuous mapping \( f : V \to V \) is strongly continuous.

Proof. Let \( \varepsilon < 1/2 \). In view of Definition 1.8, there exists \( \delta > 0 \) such that \( R_{f/\delta}(\varepsilon/2) > 1 - \varepsilon/2 \), therefore \( q \in N_p(\delta) \Rightarrow \nu_{f(q)}(\varepsilon/2) > R_{f(N_p(\delta))}(\varepsilon/2) > 1 - \varepsilon/2 \), i.e.,

\[
\nu_{p-q}(\delta) > 1 - \varepsilon \text{ implies } \nu_{f(q)}(\varepsilon/2) > 1 - \varepsilon/2.
\]

From \( p \in N_p(\delta) \), we have

\[
\nu_{f(q)}(\varepsilon/2) \geq R_{N_p(\delta)}(\varepsilon/2) > 1 - \varepsilon/2,
\]

thus

\[
\nu_{f(p)-f(q)}(\varepsilon) \geq \tau(\nu_{f(p)}, \nu_{f(q)})(\varepsilon) \\
\geq \tau_w(\nu_{f(p)}, \nu_{f(q)})(\varepsilon) \\
= \sup_{s \in \nu} W(\nu_{f(p)}(s), \nu_{f(q)}(t)) \\
\geq W(\nu_{f(p)}(\varepsilon/2), \nu_{f(q)}(\varepsilon/2)) \\
\geq W(1 - \varepsilon/2, 1 - \varepsilon/2) \\
= 1 - \varepsilon
\]

i.e., \( f(q) \in N_{f(p)}(\varepsilon) \). So \( \forall q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon) \).

Theorem 2.2. Let \((V, \nu, \tau, \tau^*)\) be a PN space, then

(i). A B-contraction mapping is strongly continuous;

(ii). an H-contraction mapping is strongly continuous.

Proof. (i). Suppose \((V, \nu, \tau, \tau^*)\) be a PN space and \( f : V \to V \) be B-contraction. According to Definition 1.6, there is a constant \( k \in (0, 1) \) such that for \( p \) and \( q \) in \( V \), and \( x > 0 \)

\[
\nu_{f(p)-f(q)}(kx) \geq \nu_{p-q}(x).
\]

Therefore, let \( a > 1 \), we have

\[
\nu_{f(p)-f(q)}(ax) \geq \nu_{f(p)-f(q)}(kx) \geq \nu_{p-q}(x).
\]
Let \( v_{p-q}(x) > 1 - x \) we have

\[
v_{f(p)-f(q)}(ax) \geq v_{p-q}(x) > 1 - x > 1 - ax,
\]

i.e.,

\[
q \in N_p(x) \Rightarrow f(q) \in N_{f(p)}(ax).
\] (11)

So for \( \varepsilon > 0 \), set \( \delta = \varepsilon / a \) such that

\[
q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon).
\] (12)

By Definition 2.1., we have that \( f \) is strongly continuous.

(ii). Suppose \((V, \nu, \tau, \tau^*)\) be a PN space and \( f : V \to V \) be H-contraction, and if \( \varepsilon > 0 \), in view of Definition 1.6, there is a constant \( k_0 \in (0,1) \) such that for \( p \) and \( q \) in \( V \),

\[
v_{p-q}(\varepsilon / k_0) > 1 - \varepsilon / k_0 \text{ implies } v_{f(p)-f(q)}(\varepsilon) > 1 - \varepsilon,
\]

i.e.,

\[
q \in N_p(\varepsilon / k_0) \Rightarrow f(q) \in N_{f(p)}(\varepsilon).
\] (14)

So for \( \varepsilon > 0 \), set \( \delta = \varepsilon / k_0 \) such that

\[
q \in N_p(\delta) \Rightarrow f(q) \in N_{f(p)}(\varepsilon).
\] (15)

Basing on Definition 2.1., we have proven that \( f \) is strongly continuous.

The following examples, Example 2.1. and 2.2., show that a B-contraction isn’t necessarily an H-contraction, an H-contraction isn’t necessarily a B-contraction, and a strongly continues mapping isn’t necessarily a B-contraction or an H-contraction.
Example 2.1. Let $V$ be a vector space and $\nu_p = \mu_p = \varepsilon_0$, if $a \in (2,3)\, , p, q \in V\, (p, q \neq \theta)$ and $x \in \overline{R}$, 

$$
\nu_p(x) = \begin{cases} 
0, x \leq a \\
1, x > a 
\end{cases} \\
\mu_p(x) = \begin{cases} 
0, x \leq 0 \\
\frac{1}{a}, 0 < x \leq \frac{2a}{3} \\
\frac{2a}{3} < x < \infty \\
1, x = \infty 
\end{cases}
$$

and if $\tau(\nu_p, \nu_q)(x) = \tau^*(\nu_p, \nu_q)(x) = \sup_{s+t=x} \min(\nu_p(s), \nu_q(t))$, then $(V, \nu, \tau, \tau^*)$ and $(V, \mu, \tau, \tau^*)$ are equilateral PN spaces by Definition 1.3. Now let $I : (V, \nu, \tau, \tau^*) \rightarrow (V, \mu, \tau, \tau^*)$ be the identity operator, then $I$ is not a $B$-contraction, but an $H$-contraction. In fact, for every $k \in (0,1)\, , x > a$ and $p \neq \theta$, 

$$
\mu_p(kx) \leq \mu_p(x) = \frac{2}{a} < 1 = \nu_p(x). \text{ Hence I is not a B-contraction.}
$$

Next we’ll prove that $I$ is an $H$-contraction. Suppose $\nu_p(x) > 1 - x$, where $p \neq \theta$. This condition holds only if $x > 1$. In fact, if $x \leq 1$, then $\nu_p(x) = 0 \leq 1 - x$. For $a \in (2,3)$, if $1 < x \leq a$, let $h = \frac{2}{3}$, then $\frac{2}{3} < hx \leq \frac{2a}{3}$, therefore $\mu_p(hx) = \mu_p(x) = \frac{1}{a} > \frac{1}{3} = 1 - \frac{2}{3} > 1 - hx$. If $x > a$, let $h = \frac{2}{3}$, then $hx > \frac{2a}{3}$, therefore $\mu_p(hx) = \mu_p(x) = \frac{2}{a} > 1 - \frac{a}{2} > 1 - \frac{2a}{3} > 1 - hx$. Thus there is a constant $h = \frac{2}{3}$ such that for all points $p \neq \theta$ in $V$, and all $x > 0$,

$$
\nu_p(x) > 1 - x \text{ implies } \mu_p(hx) > 1 - hx,
$$

i.e., $I$ is an $H$-contraction. In view of Theorem 2.2. (ii), we have that $I$ is strongly continuous.

Example 2.2. Let $V = V' = \overline{R}$, $\nu_0 = \mu_0 = \varepsilon_0$, if, for $x > 0, p \neq 0$ and $a = \frac{k+1}{2}$, where $k \in (0,1)$,
and if \( \tau(\nu_p, \nu_q)(x) = \tau^*(\nu_p, \nu_q)(x) = \text{supmin}(\nu_p(s), \nu_q(t)) \), then \((\widetilde{R}, \nu, \tau, \tau^*)\) and \((\widetilde{R}, \mu, \tau, \tau^*)\) are equilateral PN spaces by Definition 1.3. Now let \( I: (\widetilde{R}, \nu, \tau, \tau^*) \to (\widetilde{R}, \mu, \tau, \tau^*) \) be the identity operator, then \( I \) is not an H-contraction, but a B-contraction. In fact, for every \( k \in (0,1) \), we have that \( a = \frac{k+1}{2} \in (\frac{1}{2}, 2) \). Let \( x = \frac{1}{a} \), we have that \( \nu_p(x) = \nu_p(\frac{1}{a}) = \frac{1}{a} > 1 - \frac{1}{a} = 1 - x \). But,

\[
\mu_{ip}(kx) \leq \mu_{ip}(x) = \mu_{ip}(\frac{1}{a}) = \frac{1}{a} < 1 - \frac{k}{a} = 1 - kx.
\]

Hence \( I \) is not an H-contraction. Meanwhile, for every \( p \in \widetilde{R} \) and \( x > 0 \), there exists a constant \( k_0 = \frac{2}{3} \) such that

\[
\mu_{ip}(k_0 x) = \mu_{ip}(\frac{2x}{3}) = \mu_{ip}(\frac{2}{3}) \geq \mu_{ip}(\frac{x}{2}) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{a}, & 0 < x \leq a = \nu_p(x) \\ 1, & a < x \leq \infty \end{cases}
\]

i.e., \( I \) is a B-contraction. In view of Theorem 2.2(ii), \( I \) is strongly continuous.

**Example 2.3.** Let PN space \((V, v, \tau, \tau^*)\) and \((V, \mu, \tau, \tau^*)\) satisfy Example 2.1, and \( I: (V, v, \tau, \tau^*) \to (V, \mu, \tau, \tau^*) \) be the identity operator, then \( I \) is not strongly \( \varepsilon \)-continuous, but strongly continuous. In fact, according to Example 2.1., it is obvious that \( I \) is strongly continuous.

Now we are going to prove that \( I \) is not strongly \( \varepsilon \)-continuous. Suppose \( I \) is strongly \( \varepsilon \)-continuous. Let \( A \subset V \) be not empty. In view of Lemma 1.1., for each \( p \in A \) and \( \varepsilon > 0 \), we have

\[
\mu_{ip}(\varepsilon) > 1 - \varepsilon.
\]

However, let \( \varepsilon_0 \in (0, \frac{1}{3}) \), for each \( p \in A \) and \( p \neq 0 \), we have

\[
\mu_{ip}(\varepsilon_0) = \mu_{ip}(\varepsilon_0) \leq \mu_{ip}(\frac{1}{3}) = \frac{2}{3} < 1 - \varepsilon_0.
\]

Thus, there appears a contradiction. So, we have that \( I \) is not strongly \( \varepsilon \)-continuous.

**Lemma 2.1.** [10] Let \( V \) be Banach space and \( D \) be a compact and convex subset of \( V \). If \( f : D \to D \) is a strongly continuous mapping, then \( f \) has at least one fixed point on \( D \).

Not all PN spaces are Banach spaces; Lemma 2.2. shows that under some conditions, a PN space is a Banach space.
Lemma 2.2. [8] Let $(V, \nu, \tau, \tau^*)$ be a TV PN space and $N_\theta(\lambda)$ be strong $\lambda$-neighborhoods of $\theta$, where $\lambda \in (0, 1)$.

(i) Suppose $\tau \geq \tau_w$. If there is an $N_\theta(\lambda)$ satisfying $(Z_1)$, then $(V, \nu, \tau, \tau^*)$ is nomable.

(ii) Suppose $\tau \geq \tau_\pi$, $(\pi = \text{Prod})$. If there is an $N_\theta(\lambda)$ satisfying $(Z_2)$, then $(V, \nu, \tau, \tau^*)$ is nomable.

Theorem 2.3. Let $A$ be a compact and convex subset of TV PN space $(V, \nu, \tau, \tau^*)$ and $f : A \to A$ be a strongly continuous mapping.

(i) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_1)$, then $f$ has at least one fixed point on $A$.

(ii) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_2)$, then $f$ has at least one fixed point on $A$.

Proof. In view of Lemma 2.1. and Lemma 2.2., it is obvious that Theorem 2.3. holds.

Corollary 2.1. Let $A$ be a compact and convex subset of TV PN space $(V, \nu, \tau, \tau^*)$ and $f : A \to A$ be a B-contraction or an H-contraction mapping.

(i) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_1)$, then $f$ has at least one fixed point on $A$.

(ii) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_2)$, then $f$ has at least one fixed point on $A$.

Proof. In view of Theorem 2.2., we have that $f : A \to A$ is a strongly continuous mapping on $A$. By Theorem 2.3., $f$ has at least one fixed point on $A$.

Corollary 2.2. Let $A$ be a compact and convex subset of TV PN space $(V, \nu, \tau, \tau^*)$ and $f : A \to A$ be a strongly $\epsilon$-continuous mapping.

(i) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_1)$, then $f$ has at least one fixed point on $A$.

(ii) Suppose $\tau \geq \tau_w$ and there is an $N_\theta(\lambda)$ satisfying $(Z_2)$, then $f$ has at least one fixed point on $A$.

Proof. In view of Theorem 2.1., we have that $f : A \to A$ is a strongly continuous mapping on $A$. By Theorem 2.3., we have that $f$ has at least one fixed point on $A$. 369
Theorem 2.4. Let $A$ be a compact and convex subset of PN space $(V, v, \tau, \tau^*)$, where $(V, v, \tau, \tau^*)$ is a Banach space. If $f : A \to A$ is a strongly continuous mapping, then $f$ has at least one fixed point on $A$.

Proof. In view of Lemma 2.1., it is obvious that Theorem 2.4. holds.

Let $(V, v, \tau, \tau^*)$ be a PN space and $f : V \to V$ be a single-valued self mapping. A point $p \in V$ with the property $\nu_{f(p)-p} = \varepsilon_0$ is called a fixed point of $f$ on $V$. Note that, for every $p \in V \setminus \{\emptyset\}$, if $\nu_{f(p)-p}(t) < 1$ for all $t > 0$ (see [12], Example 2.4.), then $f(p) \neq p$, i.e., $f$ has no fixed point on $V$. In such a situation a question arises about the existence of an approximate fixed point. The following is the definition of the approximate fixed point in PN space.

Definition 2.2. [9] Suppose $(V, v, \tau, \tau^*)$ be a PN space and $A \subseteq V$. We call $p \in A$ an $\varepsilon$-fixed point of $f : A \to A$, if, there exists an $\varepsilon > 0$ such that $\sup_{t < \varepsilon} \nu_{f(p)-p}(t) = 1$. A self mapping $f : A \to A$ has approximate fixed point property (in short a.f.p.p.) if the function $f$ possesses at least one $\varepsilon$-fixed point.

Definition 2.3. $A$ is bounded, if for every $n \in \mathbb{N}$ and for every $p \in A$, there is a $k \in \mathbb{N}$ such that $\nu_{p/k}(1/n) > 1 - 1/n$.

Lemma 2.3. [3] If $|\alpha| \leq |\beta|$, then $\nu_{\alpha p} \geq \nu_{\beta p}$.

Theorem 2.5. Suppose $A$ be a bounded and convex subset of PN space $(V, v, \tau, \tau^*)$ with $\tau \geq \tau_w$, where $(V, v, \tau, \tau^*)$ is a Banach space. If the mapping $f : A \to A$ is strongly $\varepsilon$-continuous, then $f$ has at least one approximate fixed-point on $A$.

Proof. Since $f$ is an $\varepsilon$-continuous on $A$, by Definition 1.8. and Lemma 1.1, we have that for every $p \in A$, $\sup_{e < 0} \nu_{f(p)}(e) = 1$. Let $B$ be a compact and convex subset of $A$, defined by $B = (1-a)\overline{A}$, where $\overline{A}$ is a closure of $A$ and $(0 < a < 1)$ In view of Theorem 2.1., we have that $f$ is strongly continuous. We can define a strongly continuous function $g : B \to B$ by $g(p) = (1-a)f(p), \forall p \in B$. By Theorem 2.4., there is a $p_0 \in B$ such that $g(p_0) = p_0$, which implies $(1-a)f(p_0) = p_0$. Whence $\nu_{(1-a)f(p_0)-p_0} = \varepsilon_0$. Since $f(p_0) - p_0 = (1-a)f(p_0) - p_0 + af(p_0)$, by (PN3) and Lemma 2.3., we have
\[\nu_{f(p_0) - p_0} \geq \tau(\nu_{(1-a)f(p_0) - p_0}, \nu_{af(p_0)})
\]
\[= \tau(\nu_{p_0}, \nu_{f(p_0)})
\]
\[= \nu_{f(p_0)}.
\]

By taking sup over \(0 < t < \varepsilon\) on both sides of the inequality, we have \(\sup_{0 < t < \varepsilon} \nu_{f(p_0) - p_0} (t) \geq \sup_{0 < t < \varepsilon} \nu_{f(p_0)} (t)\).

Because \(p_0 \in B \subset A\), \(\sup_{0 < t < \varepsilon} \nu_{f(p_0)} (t) = 1\). So \(\sup_{0 < t < \varepsilon} \nu_{f(p_0) - p_0} (t) \geq \sup_{0 < t < \varepsilon} \nu_{f(p_0)} (t) = 1\). According to Definition 2.2, \(p_0\) is an approximate fixed point of \(f\), thus \(f\) has at least one \(\varepsilon\)-fixed-point on \(A\).

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**References**