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On Global Generalized Solutions for a Two-Dimensional Generalized Zakharov Equations

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Abstract

This paper considers the existence of the generalized solution to the Cauchy problem for a class of generalized Zakharov equation in two dimensions. By a priori integral estimates and Galerkin method, the author establish the existence of the global generalized solution to the problem.

Key words

Generalized Zakharov equations, Cauchy problem, generalized solution

1. Introduction

Many authors studied the Zakharov system [1-5, 8-12]. In [1], B. Guo, J. Zhang and X. Pu established globally in time existence and uniqueness of smooth solution for a generalized Zakharov equation in two dimensional case for small initial data, and proved global existence of smooth solution in one spatial dimension without any small assumption for initial data. [2] proved low-regularity global well-posedness for the 1d Zakharov system. The asymptotic behavior of Zakharov equations driven by random force is studied [3]. S. You studied a generalized Zakharov equation and obtained the existence and uniqueness of the global solutions to initial value problem [4]. Biswas and Song address the Zakharov-Kuznetsov-Benjamin-Bona-Mahoney equation with power law nonlinearity [8]. By applying the extended direct algebraic method, Seadawy founds the electric field potential, electric field and magnetic field in the form of traveling wave solutions for the two-dimensional ZK equation [9]. Adem and Muatjetjeja compute conservation laws for the 2D Zakharov-Kuznetsov equation using Noether's approach through an interesting method of increasing the order of the 2D Zakharov-Kuznetsov equation [11]. Doronin and Larkin consider initial-boundary-value problems for the linear Zakharov-Kuznetsov equation posed on bounded rectangles [12]. Special interest was recently devoted to quantum corrections to the Zakharov equations for Langmuir waves in a plasma [13]

$$iE_t - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) = nE + \Gamma\nabla\Delta(\nabla\cdot E),$$
$$n_{tt} - \Delta n = \Delta|E|^2 - \Gamma\Delta^2 n.$$

the parameter α defined as the square ratio of the light speed and the electron Fermi velocity is usually large. In contrast, the coefficient Γ that measures the influence of quantum effects is usually very small [14].

In this paper, we are interested in studying the following generalized modified Zakharov system in two dimensions

$$iE_t - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) = nE + \Gamma\nabla\Delta(\nabla\cdot E) + f(|E|^2)E, \quad (1)$$

$$n_{tt} - \Delta n = \Delta|E|^2 - \Gamma\Delta^2 n. \quad (2)$$

with initial data

$$E(x,0) = E_0(x), \quad n(x,0) = n_0(x), \quad n_t(x,0) = n_1(x). \quad (3)$$

where $E = (E_1, E_2),\ x = (x_1, x_2) \in \mathbb{R}^2$.

Now we state the main results of the paper.

Theorem 1. Suppose that

(i) $E_0(x) \in H^2(\mathbb{R}^2),\ n_0(x) \in H^1(\mathbb{R}^2),\ n_1(x) \in H^{-1}(\mathbb{R}^2)$.

(ii) $f(\xi) \in C(\mathbb{R}),\ |f(\xi)| \le M\xi^\gamma$. Where $M > 0,\ 0 \le \gamma < 1$.

Then there exists global generalized solution of the initial problem (1)-(3).

$$E(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}),$$
$$n(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}),$$
$$n_t(x,t) \in L^\infty(\mathbb{R}^+; H^{-1}) \cap W^{1,\infty}(\mathbb{R}^+; H^{-3}).$$

To study generalized solution of the system(1)-(3), we transform it into the following form (notice that $\nabla(\nabla\cdot E) = \Delta E + \nabla\times(\nabla\times E)$)

$$iE_t - (\alpha-1)\nabla\times(\nabla\times E) + \Delta E = nE + \Gamma\nabla\Delta(\nabla\cdot E) + f(|E|^2)E, \quad (4)$$

$$n_t + \nabla\cdot\varphi = 0, \quad (5)$$

$$\varphi_t = -\nabla(n + |E|^2) + \Gamma\nabla\Delta n. \quad (6)$$

with initial data

$$E(x,0)=E_0(x),\ n(x,0)=n_0(x),\ \varphi(x,0)=\varphi_0(x). \tag{7}$$

where φ_0 satisfying $-\nabla\cdot\varphi_0=n_1$.

For the sake of convenience of the following contexts, we set some notations. For $1\le q\le\infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q th power integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p=2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f,g)=\int_{\mathbb{R}^n} f(x)\cdot\overline{g(x)}dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. And we use C to represent various constants that can depend on initial data. The paper is organized as follows. In Section 2, we establish a priori estimations. In Section 3, we state the existence of global generalized solution.

2. *A priori* estimations

For the solution of system (4)-(7), we have

$$\|E(x,t)\|^2_{L^2(\mathbb{R}^2)}=\|E_0\|^2_{L^2(\mathbb{R}^2)}.$$

The conservation is obtained by taking the imaginary part of the inner product of (4) and E.

Lemma 1. Suppose that $E_0(x)\in H^2(\mathbb{R}^2),\ n_0(x)\in H^1(\mathbb{R}^2),\ \varphi_0(x)\in L^2(\mathbb{R}^2)$. Then for the solution of problem (4)-(7) we have

$$\mathrm{M}(t)=\mathrm{M}(0).$$

Where

$$\mathrm{M}(t)=\|\nabla\varepsilon\|^2_{L^2}+\int_{\mathbb{R}^2} n|\varepsilon|^2\,dx+\frac{1}{2}\|\nabla\varphi\|^2_{L^2}+\frac{1}{2}\|n\|^2_{L^2}-\frac{2\alpha}{p+2}\|\varepsilon\|^{p+2}_{L^{p+2}}.$$

Proof. Taking the inner product of (4) and E_t. Since

$$\mathrm{Re}\big(-(\alpha-1)\nabla\times(\nabla\times E),E_t\big)=-(\alpha-1)\,\mathrm{Re}\big(\nabla\times E,\nabla\times E_t\big)$$
$$=-\frac{\alpha-1}{2}\frac{d}{dt}\|\nabla\times E\|^2_{L^2},$$

$$\mathrm{Re}\big(\Delta E,E_t\big)=-\mathrm{Re}\big(\nabla E,\nabla E_t\big)=-\frac{1}{2}\frac{d}{dt}\|\nabla E\|^2_{L^2},$$

$$\mathrm{Re}\big(nE,E_t\big)=\frac{1}{2}\int n|E|^2_t\,dx=\frac{1}{2}\frac{d}{dt}\int n|E|^2\,dx-\frac{1}{2}\int n_t|E|^2\,dx,$$

$$\mathrm{Re}\big(\Gamma\nabla\Delta(\nabla\cdot E),E_t\big)=\Gamma\,\mathrm{Re}\big(\nabla(\nabla\cdot E),\nabla(\nabla\cdot E_t)\big)=\frac{\Gamma}{2}\frac{d}{dt}\|\nabla(\nabla\cdot E)\|^2_{L^2}.$$

$$\mathrm{Re}\left(f(|E|^2)E, E_t\right) = \frac{1}{2}\int f(|E|^2)|E|_t^2\,dx = \frac{1}{2}\frac{d}{dt}\int\int_0^{|E|^2} f(\xi)d\xi dx.$$

We get

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla E\|_{L^2}^2 + (\alpha-1)\|\nabla\times E\|_{L^2}^2 + \int n|E|^2\,dx\right] \\ &+\frac{d}{dt}\left[\Gamma\|\nabla(\nabla\cdot E)\|_{L^2}^2 + \int\int_0^{|E|^2} f(\xi)d\xi dx\right] = \int n_t|E|^2\,dx.\end{aligned} \tag{8}$$

From (5) and (6), we obtain

$$\begin{aligned}\int n_t|E|^2\,dx &= -\int\nabla\cdot\varphi|E|^2\,dx = \int\varphi\cdot\nabla|E|^2\,dx \\ &= \int\varphi\cdot(\Gamma\nabla\Delta n - \nabla n - \varphi_t)dx \\ &= \int\nabla\cdot\varphi(-\Gamma\Delta n + n)dx - \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 \\ &= \int n_t(\Gamma\Delta n - n)dx - \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2 \\ &= -\frac{1}{2}\frac{d}{dt}\left[\Gamma\|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2\right].\end{aligned} \tag{9}$$

Combining inequality (8) with (9) we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla E\|_{L^2}^2 + (\alpha-1)\|\nabla\times E\|_{L^2}^2 + \int n|E|^2\,dx + \Gamma\|\nabla(\nabla\cdot E)\|_{L^2}^2\right] \\ &+\frac{d}{dt}\left[\int\int_0^{|E|^2} f(\xi)d\xi dx + \frac{\Gamma}{2}\|\nabla n\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2\right] = 0.\end{aligned}$$

Thus we get Lemma 1.

Lemma 2 (Gagliardo-Nirenberg inequality [6]). Assume that $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty, 0 \le j \le m$, we have the estimations

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \le C\|D^m u\|_{L^r(\mathbb{R}^n)}^{\alpha}\|u\|_{L^q(\mathbb{R}^n)}^{1-\alpha},$$

where C is a positive constant, $0 \le \frac{j}{m} \le \alpha \le 1$, $\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$

Lemma 3. Suppose that

(i) $E_0(x) \in H^2(\mathbb{R}^2),\ n_0(x) \in H^1(\mathbb{R}^2),\ \varphi_0(x) \in L^2(\mathbb{R}^2).$

(ii) $f(\xi) \in C(\mathbb{R}),\ |f(\xi)| \le M\xi^{\gamma}$. Where $M > 0,\ 0 \le \gamma < 1.$

Then we have

$$\|\nabla E\|_{L^2}^2 + \|\nabla\times E\|_{L^2}^2 + \|\nabla(\nabla\cdot E)\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2 \le C.$$

Proof. By Hölder inequality, Young inequality and Lemma 2 we have

$$\begin{aligned}\int n\,|E|^2\,dx &\le \|n\|_{L^4}\|E\|_{L^{\frac{8}{3}}}^2 \le C\|\nabla n\|_{L^2}^{\frac{1}{2}}\|n\|_{L^2}^{\frac{1}{2}}\|\nabla E\|_{L^2}^{\frac{1}{2}}\|E\|_{L^2}^{\frac{3}{2}}\\ &\le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + C\|n\|_{L^2}^{\frac{2}{3}}\|\nabla E\|_{L^2}^{\frac{2}{3}}\\ &\le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + C\|\nabla E\|_{L^2}\\ &\le \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{4}\|\nabla E\|_{L^2}^2 + C.\end{aligned} \tag{10}$$

And noticing that $f(\xi)\in C(\mathbb{R})$, $|f(\xi)|\le M\xi^\gamma$, we get

$$\int\int_0^{|E|^2} f(\xi)d\xi dx \le \int\int_0^{|E|^2} M\xi^\gamma d\xi dx = \frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\,dx \tag{11}$$

Using Gagliardo-Nirenberg inequality and noticing that $0\le\gamma<1$, we write

$$\frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\,dx \le C\|\nabla E\|_{L^2}^{2\gamma}\|E\|_{L^2}^2 \le \frac{1}{4}\|\nabla E\|_{L^2}^2 + C. \tag{12}$$

Note that from Lemma 2 and eq. (10)-(12), one has

$$\begin{aligned}&\frac{1}{2}\|\nabla E\|_{L^2}^2 + (\alpha-1)\|\nabla\times E\|_{L^2}^2 + \Gamma\|\nabla(\nabla\cdot E)\|_{L^2}^2\\ &\quad + \frac{\Gamma}{4}\|\nabla n\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{2}\|\varphi\|_{L^2}^2 \le |\mathrm{M}(0)| + C.\end{aligned}$$

Since α is larger than 1, we thus get Lemma 3.

Lemma 4. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}^2)$, $n_0(x)\in H^1(\mathbb{R}^2)$, $\varphi_0(x)\in L^2(\mathbb{R}^2)$.

(ii) $f(\xi)\in C(\mathbb{R})$, $|f(\xi)|\le M\xi^\gamma$. Where $M>0$, $0\le\gamma<1$.

Then we have

$$\|E_t\|_{H^{-2}} + \|n_t\|_{H^{-1}} + \|\varphi_t\|_{H^{-2}} \le C.$$

Proof. Taking the inner product of eq. (4) and V, (5) and v, (6) and Φ, it follows that

$$\left(iE_t - (\alpha-1)\nabla\times(\nabla\times E) + \Delta E, V\right) = \left(nE + \Gamma\nabla\Delta(\nabla\cdot E) + f(|E|^2)E, V\right). \tag{13}$$

$$\left(n_t + \nabla\cdot\varphi, v\right) = 0, \tag{14}$$

$$\left(\varphi_t, \Phi\right) = \left(-\nabla(n+|E|^2) + \Gamma\nabla\Delta n, \Phi\right). \tag{15}$$

where $\forall\ v, v_i\in H_0^2\ (i=1,2)$, $V=(v_1,v_2)$, $\Phi=(v_1,v_2)$.

By Hölder inequality, it follows from eq. (13) that

$$
\begin{aligned}
\left|\left(E_t,V\right)\right| &\le \left|\left((\alpha-1)\nabla\times\left(\nabla\times E\right),V\right)\right| + \left|\left(\Delta E,V\right)\right| + \left|\left(nE,V\right)\right| \\
&\quad + \left|\left(\Gamma\nabla\left[\Delta\left(\nabla\cdot E\right)\right],V\right)\right| + \left|\left(f(|E|^2)E,V\right)\right| \\
&= (\alpha-1)\left|\left(\nabla\times E,\nabla\times V\right)\right| + \left|\left(\nabla E,\nabla V\right)\right| + \left|\left(nE,V\right)\right| \\
&\quad + \Gamma\left|\left(\nabla\left(\nabla\cdot E\right),\nabla\left(\nabla\cdot V\right)\right)\right| + \left|\left(f(|E|^2)E,V\right)\right| \\
&\le (\alpha-1)\left\|\nabla\times E\right\|_{L^2}\left\|\nabla\times V\right\|_{L^2} + \left\|\nabla E\right\|_{L^2}\left\|\nabla V\right\|_{L^2} + \left\|n\right\|_{L^4}\left\|E\right\|_{L^4}\left\|V\right\|_{L^2} + \\
&\quad + \Gamma\left\|\nabla\left(\nabla\cdot E\right)\right\|_{L^2}\left\|\nabla\left(\nabla\cdot V\right)\right\|_{L^2} + M\left\|E\right\|_{L^{2(2\gamma+1)}}^{2\gamma+1}\left\|V\right\|_{L^2}.
\end{aligned}
\tag{16}
$$

By Gagliardo-Nirenberg inequality, we know that

$$
\left\|E\right\|_{L^4} \le C\left\|\nabla E\right\|_{L^2}^{\frac{1}{2}}\left\|E\right\|_{L^2}^{\frac{1}{2}} \le C,
$$

$$
\left\|n\right\|_{L^4} \le C\left\|\nabla n\right\|_{L^2}^{\frac{1}{2}}\left\|n\right\|_{L^2}^{\frac{1}{2}} \le C,
$$

$$
\left\|E\right\|_{L^{2(2\gamma+1)}}^{2\gamma+1} \le C\|\nabla E\|_{L^2}^{2\gamma}\|E\|_{L^2} \le C,
$$

Hence from (16) we get

$$
\left|\left(E_t, V\right)\right| \le C\left\|V\right\|_{H_0^2}. \tag{17}
$$

Using Hölder inequality, from eq. (14), there is

$$
\left|\left(n_t, v\right)\right| = \left|\left(\nabla\cdot\varphi, v\right)\right| = \left|\left(\varphi, \nabla v\right)\right| \le \left\|\varphi\right\|_{L^2}\left\|\nabla v\right\|_{L^2} \le C\left\|v\right\|_{H_0^1}, \tag{18}
$$

From eq. (15) and Hölder inequality, we have

$$
\begin{aligned}
\left|\left(\varphi_t,\Phi\right)\right| &= \left|\left(\nabla n,\Phi\right)\right| + \left|\left(\nabla|E|^2,\Phi\right)\right| + \left|\left(\Gamma\nabla\Delta n,\Phi\right)\right| \\
&\le \left\|\nabla n\right\|_{L^2}\left\|\Phi\right\|_{L^2} + \left|\left(|E|^2,\nabla\cdot\Phi\right)\right| + \Gamma\left|\left(\nabla n,\Delta\Phi\right)\right| \\
&\le \left\|\nabla n\right\|_{L^2}\left\|\Phi\right\|_{L^2} + \left\|E\right\|_{L^4}^2\left\|\nabla\cdot\Phi\right\|_{L^2} + \Gamma\left\|\nabla n\right\|_{L^2}\left\|\Delta\Phi\right\|_{L^2} \\
&\le C\left\|\Phi\right\|_{H_0^2}.
\end{aligned}
\tag{19}
$$

Hence from (17)-(19), one obtain Lemma 4.

3. The existence of generalized solution

In this section, we formulate the proof of Theorem 1. First we give the definition of generalized solution for problem (4)-(7).

Definition 1. The functions $E(x,t)\in L^{\infty}(\mathbb{R}^{+};H^{1})\cap W^{1,\infty}(\mathbb{R}^{+};H^{-2})$, $n(x,t)\in L^{\infty}(\mathbb{R}^{+};H^{1})\cap W^{1,\infty}(\mathbb{R}^{+};H^{-1})$, $\varphi(x,t)\in L^{\infty}(\mathbb{R}^{+};L^{2})\cap W^{1,\infty}(\mathbb{R}^{+};H^{-2})$, are called generalized solution of problem (4)-(7), if for any they satisfy the integral equality

$$\begin{aligned}
&\left(iE_{mt},v\right)+(\alpha-1)\sum_{\substack{v\neq m\\ v\in\{1,2\}}}\left(\frac{\partial E_{v}}{\partial x_{m}},\frac{\partial v}{\partial x_{v}}\right)-(\alpha-1)\sum_{\substack{v\neq m\\ v\in\{1,2\}}}\left(\frac{\partial E_{m}}{\partial x_{v}},\frac{\partial v}{\partial x_{v}}\right)-\left(\nabla E_{m},\nabla v\right)\\
&=\left(nE_{m},v\right)+\Gamma\left(\nabla\left(\nabla\cdot E\right),\nabla\left(\frac{\partial v}{\partial x_{m}}\right)\right)+\left(f(|E|^{2})E_{m},v\right),\quad m=1,2,\\
&\left(n_{t}+\nabla\cdot\varphi,v\right)=0,\\
&\left(\varphi_{\lambda t},v\right)=\left(-\frac{\partial(n+|E|^{2})}{\partial x_{\lambda}}+\Gamma\frac{\partial\left(\Delta n\right)}{\partial x_{\lambda}},v\right),\quad \lambda=1,2.
\end{aligned}$$

with initial data

$$E|_{t=0}=E_{0}(x),\ n|_{t=0}=n_{0}(x),\ \varphi|_{t=0}=\varphi_{0}(x),$$

Next, we give two lemmas recalled in [7].

Lemma 5. Let B_0, B, B_1 be three reflexive Banach spaces and assume that the embedding $B_0\to B$ is compact. Let

$$W=\left\{V\in L^{p_0}((0,T);B_0),\frac{\partial V}{\partial t}\in L^{p_1}((0,T);B_1)\right\},\quad T<\infty,1<p_0,p_1<\infty.$$

W is a Banach space with norm

$$\|V\|_{W}=\|V\|_{L^{p_0}((0,T);B_0)}+\|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W\to L^{p_0}((0,T);B)$ is compact.

Lemma 6. Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon\in L^p(\mathbb{R}^n)$, $1<p<\infty$, such that

$$g_\varepsilon\to g\quad\text{a.e. in }\Omega\qquad\text{and}\qquad \|g_\varepsilon\|_{L^p(\Omega)}\leq C.$$

Then $g_\varepsilon\to g$ weakly in $L^p(\Omega)$.

Now, one can estimate the following theorem.

Theorem 2. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}^2),\ n_0(x)\in H^1(\mathbb{R}^2),\ \varphi_0(x)\in L^2(\mathbb{R}^2).$

(ii) $f(\xi) \in C(\mathbb{R})$, $|f(\xi)| \le M\xi^{\gamma}$. Where $M > 0,\ 0 \le \gamma < 1$.

Then there exists global generalized solution of the initial value problem (4)-(7).

$$
\begin{aligned}
&E(x,t) \in L^{\infty}(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}),\\
&n(x,t) \in L^{\infty}(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}),\\
&\varphi(x,t) \in L^{\infty}(\mathbb{R}^+; L^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}).
\end{aligned}
$$

Proof. By using Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$
-\Delta\omega_j(x) = \lambda_j \omega_j(x),\ \ \omega_j(x) \in H_0^2(\Omega),\ \ j = 1,2,\cdots,l.
$$

The approximate solution of problem (4)-(7) can be written as

$$
E^l(x,t) = \sum_{j=1}^{l} \alpha_j^l(t)\omega_j(x),\quad \varphi^l(x,t) = \sum_{j=1}^{l} \beta_j^l(t)\omega_j(x),\quad n^l(x,t) = \sum_{j=1}^{l} \gamma_j^l(t)\omega_j(x),
$$

where

$$
E^l = \left(E_1^l, E_2^l\right),\quad \alpha_j^l(t) = (\alpha_{j1}^l(t), \alpha_{j2}^l(t)),
$$

$$
\varphi^l = \left(\varphi_1^l, \varphi_2^l\right),\quad \beta_j(t) = (\beta_{j1}(t), \beta_{j2}(t)).
$$

and Ω is a 2-dimensional cube with $2D$ in each direction, that is, $\bar{\Omega} = \{x = (x_1, x_2) \,\|\, x_i| \le 2D, i = 1,2\}$. According to Galerkin's method, these undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations

$$
\begin{aligned}
&\left(iE_{mt}^l, \omega_\kappa\right) + (\alpha - 1)\sum_{\substack{\nu \ne m\\ \nu\in\{1,2\}}} \left(\frac{\partial E_\nu^l}{\partial x_m}, \frac{\partial \omega_\kappa}{\partial x_\nu}\right) - (\alpha - 1)\sum_{\substack{\nu \ne m\\ \nu\in\{1,2\}}} \left(\frac{\partial E_m^l}{\partial x_\nu}, \frac{\partial \omega_\kappa}{\partial x_\nu}\right) - \left(\nabla E_m^l, \nabla\omega_\kappa\right)\\
&= \left(nE_m^l, \omega_\kappa\right) + \Gamma\left(\nabla\left(\nabla\cdot E^l\right), \nabla\left(\frac{\partial\omega_\kappa}{\partial x_m}\right)\right) + \left(f(|E^l|^2)E_m^l, \omega_\kappa\right),\quad m = 1,2,
\end{aligned}
\tag{20}
$$

$$
\left(n_t^l + \nabla\cdot\varphi^l, \omega_\kappa\right) = 0,\quad \kappa = 1,2,\cdots,l, \tag{21}
$$

$$
\left(\varphi_{\lambda t}^l, \omega_\kappa\right) = \left(-\frac{\partial(n^l + |E^l|^2)}{\partial x_\lambda} + \Gamma\frac{\partial\left(\Delta n^l\right)}{\partial x_\lambda}, \omega_\kappa\right),\quad \lambda = 1,2, \tag{22}
$$

with initial data

$$
E^l|_{t=0} = E_0^l(x),\quad n^l|_{t=0} = n_0^l(x),\quad \varphi^l|_{t=0} = \varphi_0^l(x). \tag{23}
$$

Suppose

$$
E_0^l(x) \xrightarrow{H^1} E_0(x),\quad n_0^l(x) \xrightarrow{H^1} n_0(x),\quad \varphi_0^l(x) \xrightarrow{L^2} \varphi_0(x),\quad l \to \infty.
$$

Similarly to the proof of Lemma 1-4, for the solution $E^l(x,t)$, $n^l(x,t)$, $\varphi^l(x,t)$ of problem (20)-(23), we can establish the following estimations

$$\left\|\nabla\times E^l\right\|_{L^2}^2+\left\|E^l\right\|_{H^1}^2+\left\|\nabla\left(\nabla\cdot E^l\right)\right\|_{L^2}^2+\left\|\varphi^l\right\|_{L^2}^2+\left\|n^l\right\|_{H^1}^2\le C, \tag{24}$$

$$\left\|E_t^l\right\|_{H^{-2}}+\left\|\varphi_t^l\right\|_{H^{-2}}+\left\|n_t^l\right\|_{H^{-1}}\le C. \tag{25}$$

where the constant C is independent of l and D. By compact argument, some subsequence of $\left(E^l, n^l, \varphi^l\right)$, also labeled by l, has a weak limit $\left(E, n, \varphi\right)$. More precisely

$$E^l\to E \quad \text{in} \quad L^\infty(\mathbb{R}^+;H^1) \quad \text{weakly star,} \tag{26}$$

$$n^l\to n \quad \text{in} \quad L^\infty(\mathbb{R}^+;H^1) \quad \text{weakly star,} \tag{27}$$

$$\varphi^l\to \varphi \quad \text{in} \quad L^\infty(\mathbb{R}^+;L^2) \quad \text{weakly star.}$$

Eq. (25) imply that

$$E_t^l\to E_t \quad \text{in} \quad L^\infty(\mathbb{R}^+,H^{-2}) \quad \text{weakly star,} \tag{28}$$

$$n_t^l\to n_t \quad \text{in} \quad L^\infty(\mathbb{R}^+,H^{-1}) \quad \text{weakly star,}$$

$$\varphi_t^l\to \varphi_t \quad \text{in} \quad L^\infty(\mathbb{R}^+,H^{-2}) \quad \text{weakly star.}$$

Moreover, let us note that the following maps are continuous.

$$H^1(\mathbb{R}^2)\to L^4(\mathbb{R}^2), \quad u\mapsto u,$$

$$H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\to L^2(\mathbb{R}^2), \quad (u,v)\mapsto uv.$$

It then follows from eq. (26) and (27) that

$$\left|E^l\right|^2\to w \quad \text{in} \quad L^\infty(\mathbb{R}^+,L^2) \quad \text{weakly star,} \tag{29}$$

$$n^lE^l\to z \quad \text{in} \quad L^\infty(\mathbb{R}^+,L^2) \quad \text{weakly star.} \tag{30}$$

First, we prove $w=\left|E\right|^2$. Let Ω be any bounded subdomain of $\mathbb{R}^2$. We notice that

the embedding $H^1(\Omega)\to L^4(\Omega)$ is compact,

and for any Banach space X,

the embedding $L^\infty(\mathbb{R}^+,X)\to L^2(0,T;X)$ is continuous.

Hence, according to eq. (26), (28) and Lemma 5, applied to $B_0=H^1(\Omega), B=L^4(\Omega), B_1=H^{-2}(\Omega)$, and says that some subsequence of $E^l|_\Omega$ (also labeled by l) converges strongly to $E|_\Omega$ in $L^2(0,T;L^4(\Omega))$. So we can assume that

$$E^l\to E \quad \text{strongly in} \quad L^2(0,T;L_{loc}^4(\Omega)), \tag{31}$$

and thus

$$E^l \to E \quad \text{a.e. in} \quad [0,T]\times\Omega.$$

Then, using Lemma 6 and eq. (29) imply that $w = |E|^2$.

Second, we prove $z = nE$. Let ψ be some test function in $L^2(0,T;H^1)$, $\operatorname{supp}\psi \subset \Omega \subset \mathbb{R}^2$.

$$\int_0^T \int_{\mathbb{R}^2} (n^l E^l - nE)\psi\, dxdt = \int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt + \int_0^T \int_\Omega \left(n^l - n\right) E\psi\, dxdt.$$

On one hand

$$\left|\int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt\right| \le \left\|n^l\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))} \left\|\psi\right\|_{L^2(0,T;L^4(\Omega))}.$$

Since Ω is bounded, we deduce from eq. (27) and (31) that

$$\int_0^T \int_\Omega n^l \left(E^l - E\right)\psi\, dxdt \to 0 \quad (l \to +\infty).$$

On the other hand, let us note that $E\psi \in L^1(0,T;L^2)$. In fact

$$\left\|E\psi\right\|_{L^1(0,T;L^2)} \le \left\|E\right\|_{L^2(0,T;L^4)} \left\|\psi\right\|_{L^2(0,T;L^4)} < \infty.$$

Therefore, we deduce from eq. (27) that

$$\int_0^T \int_\Omega (n^l - n) E\psi\, dxdt \to 0 \quad (l \to +\infty).$$

Thus $n^l E^l \to nE$ in $L^2(0,T;H^{-1})$. So $z = nE$.

Hence taking $l \to \infty$ from eq. (20)-(25), by using the density of ω_j in $H_0^2(\Omega)$ we get the existence of local generalized solution for the periodic initial value problem (4)-(7). letting $D \to \infty$, the existence of local solution for the initial value problem (4)-(7) can be obtain. By the continuation extension principle and a prior estimate we can get the existence of global generalized solution for problem (4)-(7).

We thus complete the proof of Theorem 2. Hence one can get Theorem 1.

Conclusion

This paper considers the existence of the generalized solution to the Cauchy problem for a generalized Zakharov equation in two dimensions by a priori integral estimates and Galerkin method, one has the existence of the global generalized solution to the problem.

Discussion

One can regard (1)-(2) as the Langmuir turbulence parameterized by $\Gamma\ (0<\Gamma<1)$ and study the asymptotic behavior of the systems (1)-(2) when Γ goes to zero.

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