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On Global Generalized Solution for a Generalized Zakharov Equations

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Abstract

This paper considers the existence of the generalized solution to the initial vale problem for a class of generalized Zakharov equation in dimension one. by a priori integral estimates and Galerkin method, one has the existence of the global generalized solution to the problem. The obtained results may be useful for better understanding this generalized Zakharov equation.

Key words

Generalized Zakharov equations, initial value problem, generalized solution

1. Introduction

As well known, in the ineracion of laser-plasma, Zkaharov equation play a important role. Zakharov equations, derived by Zakharov in 1972 [1]. This system attracted many scientists' wide interest and attention [2-9, 12, 13]. S. You studied a generalized Zakharov equation and obtained the existence and uniqueness of the global solutions to initial value problem [14]. Recently, a quantum modified Zakharov system was derived, by means of the quantum plasma hydrodynamic model.

$$iE_t + E_{xx} - H^2 E_{xxxx} = nE,$$
$$n_{tt} - n_{xx} + H^2 n_{xxxx} = |E|^2_{xx},$$

where $E:\square^{d+1}\rightarrow\square^{d}$ is the slowly varying amplitude of the high-frequency electric field, and $n:\square^{d+1}\rightarrow\square$ denotes the fluctuation of the ion-density from its equilibrium. H is the dimensionless quantum parameter given by the ratio of the ion plasmon and electron thermal energies.

In this paper, we are interested in studying the following generalized Zakharov system.

$$iE_t + E_{xx} - H^2 E_{xxxx} = nE + f(|E|^2)E, \tag{1}$$

$$n_{tt} - n_{xx} + H^2 n_{xxxx} = |E|^2_{xx}, \tag{2}$$

with initial data

$$E(x,0) = E_0(x), \quad n(x,0) = n_0(x), \quad n_t(x,0) = n_1(x). \tag{3}$$

where $E(x,t) = \left(E_1(x,t), E_2(x,t), \cdots, E_N(x,t)\right)$ is an N -dimensional complex valued unknown functional vector, $n(x,t)$ is a real-valued unknown function, $x \in \mathbb{R}$.

Now we state the main results of the paper.

Theorem 1. Suppose that

(i) $E_0(x) \in H^2(\mathbb{R})$, $n_0(x) \in H^1(\mathbb{R})$, $n_1(x) \in H^{-1}(\mathbb{R})$.

(ii) $f(\xi) \in C(\mathbb{R})$, $|f(\xi)| \le M\xi^\gamma$. Where $M > 0$, $0 \le \gamma < 4$.

Then there exists global generalized solution of the Cauchy problem (1)-(3).

$$\begin{aligned} &E(x,t) \in L^\infty(\mathbb{R}^+; H^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2}), \\ &n(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1}), \\ &n_t(x,t) \in L^\infty(\mathbb{R}^+; H^{-1}) \cap W^{1,\infty}(\mathbb{R}^+; H^{-3}). \end{aligned}$$

To study generalized solution of the system (1)-(3), we transform it into the following form

$$iE_t + E_{xx} - H^2 E_{xxxx} = nE + f(|E|^2)E, \tag{4}$$

$$n_t - \varphi_{xx} = 0, \tag{5}$$

$$\varphi_t - n + H^2 n_{xx} - |E|^2 = 0, \tag{6}$$

with initial data

$$E(x,0) = E_0(x), \; n(x,0) = n_0(x), \; \varphi(x,0) = \varphi_0(x). \tag{7}$$

where φ_0 satisfying $\varphi_{0xx} = n_1$.

For the sake of convenience of the following contexts, we set some notations. For $1 \le q \le \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q times integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p = 2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f,g) = \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$. And we use C to represent various constants that can depend on initial data.

The paper is organized as follows. In Section 2, we make a priori estimates of the problem (4)-(7). In Section 3, we obtain the existence and uniqueness of the global generalized solution of the problem (1)-(3) by Galerkin method.

2. *A priori* estimates

In this section, we will derive a priori estimates for the solution of the system (4)-(7).

Lemma 1. Suppose that $E_0(x) \in L^2(\mathbb{R})$. Then for the solution of problem(4)-(7) we have

$$\|E(x,t)\|_{L^2(\mathbb{R})}^2 = \|E_0\|_{L^2(\mathbb{R})}^2 .$$

Proof. Taking the inner product of (4) and E. Since

$$\operatorname{Im}\left(iE_t, E\right) = \operatorname{Re}\left(E_t, E\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2}^2 ,$$

$$\operatorname{Im}\left(E_{xx} - H^2 E_{xxxx},\ E\right) = 0.$$

$$\operatorname{Im}\left(nE + f(|E|^2)E, E\right) = 0.$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E(x,t)\|_{L^2}^2 = 0,$$

we thus get Lemma 1.

Lemma 2. Suppose that $E_0(x) \in H^2(\mathbb{R})$, $\varphi_0(x) \in H^1(\mathbb{R})$, $n_0(x) \in H^1(\mathbb{R})$. Then we have

$$\mathrm{H}(t) = \mathrm{H}(0).$$

where

$$\mathrm{H}(t) = \|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\mathbb{R}} n|E|^2\,\mathrm{d}x + \int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x$$
$$+\frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|n_x\|_{L^2}^2 + \frac{1}{2}\|\varphi_x\|_{L^2}^2 ,$$

Proof. Taking the inner products of (4) and $-E_t$. Since

$$\operatorname{Re}\left(iE_t, -E_t\right) = \operatorname{Im}\left(iE_t, E_t\right) = 0,$$

$$\operatorname{Re}\left(E_{xx}, -E_t\right) = \operatorname{Re}\left(E_x, E_{xt}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_x\|_{L^2}^2 ,$$

$$\operatorname{Re}\left(-H^2E_{xxxx},\ -E_t\right) = H^2\operatorname{Re}\left(E_{xx},\ E_{xxt}\right) = \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_{xx}\|_{L^2}^2 ,$$

$$\operatorname{Re}\left(nE, -E_t\right) = -\frac{1}{2}\int_{\mathbb{R}} n|E|_t^2\,\mathrm{d}x$$
$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} n|E|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} n_t|E|^2\,\mathrm{d}x,$$

$$\mathrm{Re}\left(f(|E|^2)E, -E_t\right) = -\frac{1}{2}\int_{\mathbb{R}} f(|E|^2)|E|_t^2\,\mathrm{d}x$$
$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x.$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\mathbb{R}} n|E|^2\,\mathrm{d}x + \int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x\right] = \int_{\mathbb{R}} n_t|E|^2\,\mathrm{d}x. \tag{8}$$

Taking the inner products of (6) and n_t. Since

$$\left(-n, n_t\right) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2,$$

$$\left(H^2 n_{xx}, n_t\right) = -H^2\left(n_x, n_{xt}\right) = -\frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2,$$

it follows that

$$\int_{\mathbb{R}} \varphi_t n_t \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2 + \frac{H^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n_x\|_{L^2}^2 + \int_{\mathbb{R}} |E|^2 n_t \mathrm{d}x \tag{9}$$

Taking the inner products of (5) and φ_t. Since

$$\left(-\varphi_{xx}, \varphi_t\right) = \left(\varphi_x, \varphi_{xt}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x\|_{L^2}^2$$

it follows that

$$\int_{\mathbb{R}} \varphi_t n_t dx + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x\|_{L^2}^2 = 0 \tag{10}$$

Hence from (8)-(10) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\mathbb{R}} n|E|^2\,\mathrm{d}x + \int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x\right]$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|n_x\|_{L^2}^2 + \frac{1}{2}\|\varphi_x\|_{L^2}^2\right] = 0.$$

Letting

$$\mathrm{H}(t) = \|E_x\|_{L^2}^2 + H^2\|E_{xx}\|_{L^2}^2 + \int_{\mathbb{R}} n|E|^2\,\mathrm{d}x + \int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x$$
$$+\frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|n_x\|_{L^2}^2 + \frac{1}{2}\|\varphi_x\|_{L^2}^2,$$

It follows that

$$\mathrm{H}(t) = \mathrm{H}(0).$$

Lemma 3 (Gagliardo-Nirenberg inequality [10]). Assume that $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty, 0 \le j \le m$, we have the estimations

$$\left\|D^j u\right\|_{L^p(\mathbb{R}^n)} \le C\left\|D^m u\right\|^{\alpha}_{L^r(\mathbb{R}^n)} \left\|u\right\|^{1-\alpha}_{L^q(\mathbb{R}^n)},$$

where C is a positive constant, $0 \le \frac{j}{m} \le \alpha \le 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$$

Lemma 4. Suppose that

(i) $E_0(x) \in H^2(\mathbb{R})$, $n_0(x) \in H^1(\mathbb{R})$, $\varphi_0(x) \in H^1(\mathbb{R})$.

(ii) $f(\xi) \in C(\mathbb{R})$, $|f(\xi)| \le M\xi^{\gamma}$. Where $M > 0$, $0 \le \gamma < 4$.

Then we have

$$\left\|E_x\right\|^2_{L^2} + \left\|E_{xx}\right\|^2_{L^2} + \left\|n\right\|^2_{L^2} + \left\|n_x\right\|^2_{L^2} + \left\|\varphi_x\right\|^2_{L^2} \le C.$$

Proof. By Hölder inequality, Young inequality, there holds

$$\left|\int_{\mathbb{R}} n\left|E\right|^2 dx\right| \le \left\|n\right\|_{L^2} \left\|E\right\|^2_{L^4} \le \frac{1}{4}\left\|n\right\|^2_{L^2} + \left\|E\right\|^4_{L^4}. \tag{11}$$

using Gagliardo-Nirenberg inequality and Young inequality, we write

$$\left\|E\right\|^4_{L^4} \le C\left\|E_x\right\|_{L^2} \left\|E\right\|^3_{L^2} \le \frac{1}{2}\left\|E_x\right\|^2_{L^2} + C. \tag{12}$$

And noticing that $f(\xi) \in C(\mathbb{R})$, $|f(\xi)| \le M\xi^{\gamma}$, we get

$$\left|\int\int_0^{|E|^2} f(\xi)\mathrm{d}\xi\mathrm{d}x\right| \le \int\int_0^{|E|^2} M\xi^{\gamma}\mathrm{d}\xi\mathrm{d}x = \frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\,\mathrm{d}x \tag{13}$$

Using Gagliardo-Nirenberg inequality and noticing that $0 \le \gamma < 4$, we write

$$\frac{M}{\gamma+1}\int |E|^{2(\gamma+1)}\,dx \le C\left\|E_{xx}\right\|^{\frac{\gamma}{2}}_{L^2} \left\|E\right\|^{\frac{3\gamma+4}{2}}_{L^2} \le \frac{H^2}{2}\left\|E_{xx}\right\|^2_{L^2} + C. \tag{14}$$

Note that from Lemma 2 and eq. (11)-(14) , one has

$$\frac{1}{2}\left\|E_x\right\|^2_{L^2} + \frac{H^2}{2}\left\|E_{xx}\right\|^2_{L^2} + \frac{1}{4}\left\|n\right\|^2_{L^2} + \frac{H^2}{2}\left\|n_x\right\|^2_{L^2} + \frac{1}{2}\left\|\varphi_x\right\|^2_{L^2} \le \left|\mathrm{H}(0)\right| + C.$$

we thus get Lemma 4.

Lemma 5. Suppose that the conditions of Lemma 4 are satisfied. Then we have

$$\left\|E_t\right\|_{H^{-2}} + \left\|n_t\right\|_{H^{-1}} + \left\|\varphi_t\right\|_{H^{-1}} \le C.$$

Proof. Taking the inner product of eq. (4) and Φ, eq. (5) and ζ, eq. (6) and ζ, it follows that

$$\left(iE_t + E_{xx} - H^2 E_{xxxx},\ \Phi\right) = \left(nE + f(|E|^2)E,\ \Phi\right) \tag{15}$$

$$\left(n_t - \varphi_{xx},\ \zeta\right) = 0 \tag{16}$$

$$\left(\varphi_t - n + H^2 n_{xx} - |E|^2,\ \zeta\right) = 0. \tag{17}$$

where $\forall\ \zeta,\ \zeta_i \in H_0^2\ (i = 1, \cdots, N)$, $\Phi = (\zeta_1, \cdots, \zeta_N)$.

By Hölder inequality and Gagliardo-Nirenberg inequality, it follows from eq. (15) that

$$\begin{aligned}
\left|\left(E_t, \Phi\right)\right| &\le \left|\left(E_{xx}, \Phi\right)\right| + \left|\left(H^2 E_{xxxx}, \Phi\right)\right| + \left|\left(nE, \Phi\right)\right| + \left|\left(f(|E|^2)E,\ \Phi\right)\right| \\
&= \left|\left(E_{xx}, \Phi\right)\right| + H^2\left|\left(E_{xx}, \Phi_{xx}\right)\right| + \left|\left(nE, \Phi\right)\right| + \left|\left(f(|E|^2)E,\ \Phi\right)\right| \\
&\le \|E_{xx}\|_{L^2}\|\Phi\|_{L^2} + H^2\|E_{xx}\|_{L^2}\|\Phi_{xx}\|_{L^2} + \|n\|_{L^4}\|E\|_{L^4}\|\Phi\|_{L^2} + M\|E\|_{L^{2(2\gamma+1)}}^{2\gamma+1}\|\Phi\|_{L^2} \\
&\le C\|\Phi\|_{L^2} + C\|\Phi_{xx}\|_{L^2} + C\|n_x\|_{L^2}^{\frac{1}{4}}\|n\|_{L^2}^{\frac{3}{4}}\|E_x\|_{L^2}^{\frac{1}{4}}\|E\|_{L^2}^{\frac{3}{4}}\|\Phi\|_{L^2} + C\|E_x\|_{L^2}^{\gamma}\|E\|_{L^2}^{\gamma+1}\|\Phi\|_{L^2} \\
&\le C\|\Phi\|_{H_0^2}
\end{aligned} \tag{18}$$

Using Hölder inequality, from eq.(16), there is

$$\left|\left(n_t, \zeta\right)\right| = \left|\left(\varphi_{xx}, \zeta\right)\right| = \left|\left(\varphi_x, \zeta_x\right)\right| \le \|\varphi_x\|_{L^2}\|\zeta_x\|_{L^2} \le C\|\zeta\|_{H_0^1} \tag{19}$$

From eq. (17) and Hölder inequality, we have

$$\begin{aligned}
\left|\left(\varphi_t,\ \zeta\right)\right| &\le \left|\left(n,\ \zeta\right)\right| + \left|\left(H^2 n_{xx},\ \zeta\right)\right| + \left|\left(|E|^2,\ \zeta\right)\right| \\
&= \left|\left(n,\ \zeta\right)\right| + H^2\left|\left(n_x,\ \zeta_x\right)\right| + \left|\left(|E|^2,\ \zeta\right)\right| \\
&\le \|n\|_{L^2}\|\zeta\|_{L^2} + H^2\|n_x\|_{L^2}\|\zeta_x\|_{L^2} + \|E\|_{L^4}^2\|\zeta\|_{L^2} \\
&\le C\|\zeta\|_{L^2} + C\|\zeta_x\|_{L^2} + C\|E_x\|_{L^2}^{\frac{1}{2}}\|E\|_{L^2}^{\frac{3}{2}}\|\zeta\|_{L^2} \\
&\le C\|\xi\|_{H_0^1}.
\end{aligned} \tag{20}$$

Hence from(18)-(20), one obtain Lemma 5.

3. The existence of generalized solution

In this section, we formulate the proof of Theorem 1. First we give the definition of generalized solution for problem (4)-(7).

Definition 1. The functions $E(x,t) \in L^\infty(\mathbb{R}^+; H^2) \cap W^{1,\infty}(\mathbb{R}^+; H^{-2})$, $n(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1})$, $\varphi(x,t) \in L^\infty(\mathbb{R}^+; H^1) \cap W^{1,\infty}(\mathbb{R}^+; H^{-1})$ are called generalized solution of problem (4)-(7), if they satisfy the following integral equality

$$\begin{aligned}&\left(iE_{mt},\zeta\right)+\left(E_{mxx},\zeta\right)-H^2\left(E_{mxx},\zeta_{xx}\right)\\&\quad=\left(nE_m,\zeta\right)+\left(f(|E|^2)E_m,\zeta\right),\ m=1,\cdots,N,\end{aligned}$$

$$\left(n_t,\ \zeta\right)+\left(\varphi_x,\ \zeta_x\right)=0,$$

$$\left(\varphi_t,\ \zeta\right)-\left(n,\ \zeta\right)-H^2\left(n_x,\ \zeta_x\right)-\left(|E|^2,\ \zeta\right)=0.$$

with initial data

$$E|_{t=0}=E_0(x),\quad n|_{t=0}=n_0(x),\quad \varphi|_{t=0}=\varphi_0(x),$$

Next, we give two lemmas recalled in [11].

Lemma 6. Let B_0, B, B_1 be three reflexive Banach spaces and assume that the embedding $B_0\to B$ is compact. Let

$$W=\left\{V\in L^{p_0}((0,T);B_0),\frac{\partial V}{\partial t}\in L^{p_1}((0,T);B_1)\right\},\quad T<\infty, 1<p_0,p_1<\infty.$$

W is a Banach space with norm

$$\|V\|_W=\|V\|_{L^{p_0}((0,T);B_0)}+\|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W\to L^{p_0}((0,T);B)$ is compact.

Lemma 7. Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon\in L^p(\mathbb{R}^n)$, $1<p<\infty$, such that

$$g_\varepsilon\to g\quad \text{a.e. in }\Omega\qquad\text{and}\qquad \|g_\varepsilon\|_{L^p(\Omega)}\le C.$$

Then $g_\varepsilon\to g$ weakly in $L^p(\Omega)$.

Now, one can estimate the following theorem.

Theorem 2. Suppose that

(i) $E_0(x)\in H^2(\mathbb{R}),\ n_0(x)\in H^1(\mathbb{R}),\ \varphi_0(x)\in H^1(\mathbb{R})$.

(ii) $f(\xi)\in C(\mathbb{R}),\ |f(\xi)|\le M\xi^\gamma$. Where $M>0,\ 0\le\gamma<4$.

Then there exists global generalized solution of the initial value problem (4)-(7).

$$\begin{aligned}&E(x,t)\in L^\infty(\mathbb{R}^+;H^2)\cap W^{1,\infty}(\mathbb{R}^+;H^{-2}),\\&n(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H^{-1}),\\&\varphi(x,t)\in L^\infty(\mathbb{R}^+;H^1)\cap W^{1,\infty}(\mathbb{R}^+;H^{-1}).\end{aligned}$$

Proof. By using Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$-\Delta\omega_j(x)=\lambda_j\omega_j(x),\ \omega_j(x)\in H_0^2(\Omega),\ j=1,2,\cdots,l.$$

The approximate solution of problem (4)-(7) can be written as

$$E^l(x,t)=\sum_{j=1}^{l}\alpha_j^l(t)\omega_j(x),\quad n^l(x,t)=\sum_{j=1}^{l}\beta_j^l(t)\omega_j(x),\quad \varphi^l(x,t)=\sum_{j=1}^{l}\gamma_j^l(t)\omega_j(x),$$

where

$$E^l=\left(E_1^l,\cdots,E_N^l\right),\quad \alpha_j^l(t)=\left(\alpha_{j1}^l(t),\cdots,\alpha_{jN}^l(t)\right),$$

and Ω is a 1-dimensional cube with $2D$ in each direction, that is, $\bar{\Omega}=\{x\,||\,x|\le 2D\}$. According to Galerkin's method, these undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations

$$\begin{aligned}&\left(iE_{mt}^l,\omega_\kappa\right)+\left(E_{mxx}^l,\omega_\kappa\right)-H^2\left(E_{mxx}^l,\omega_{\kappa xx}\right)\\&\quad=\left(nE_m^l,\omega_\kappa\right)+\left(f(|E^l|^2)E_m^l,\omega_\kappa\right),\ m=1,\cdots,N,\end{aligned}\tag{21}$$

$$\left(n_t^l,\omega_\kappa\right)+\left(\varphi_x^l,\omega_{\kappa x}\right)=0,\quad \kappa=1,2,\cdots,l,\tag{22}$$

$$\left(\varphi_t^l,\omega_\kappa\right)-\left(n^l,\omega_\kappa\right)-H^2\left(n_x^l,\omega_{\kappa x}\right)-\left(|E^l|^2,\ \omega_\kappa\right)=0.\tag{23}$$

with initial data

$$E^l|_{t=0}=E_0^l(x),\quad n^l|_{t=0}=n_0^l(x),\quad \varphi^l|_{t=0}=\varphi_0^l(x),\tag{24}$$

Suppose

$$E_0^l(x)\xrightarrow{H^2}E_0(x),\quad n_0^l(x)\xrightarrow{H^1}n_0(x),\quad \varphi_0^l(x)\xrightarrow{H^1}\varphi_0(x),\quad l\to\infty.$$

Similarly to the proof of lemma 1-5, for the solution $E^l(x,t)$, $n^l(x,t)$, $\varphi^l(x,t)$ of problem (21)-(24), we can establish the following estimations

$$\left\|E^l\right\|_{H^2}+\left\|n^l\right\|_{H^1}+\left\|\varphi^l\right\|_{H^1}\le C,\tag{25}$$

$$\left\|E_t^l\right\|_{H^{-2}}+\left\|n_t^l\right\|_{H^{-1}}+\left\|\varphi_t^l\right\|_{H^{-1}}\le C.\tag{26}$$

where the constant C is independent of l and D. By compact argument, some subsequence of $\left(E^l,\ n^l,\ \varphi^l\right)$, also labeled by l, has a weak limit $\left(E,\ n,\ \varphi\right)$. More precisely

$$E^l\to E\quad\text{in}\quad L^\infty(\mathbb{R}^+;H^2)\quad\text{weakly star},\tag{27}$$

$$n^l\to n\quad\text{in}\quad L^\infty(\mathbb{R}^+;H^1)\quad\text{weakly star},\tag{28}$$

$$\varphi^l\to\varphi\quad\text{in}\quad L^\infty(\mathbb{R}^+;H^1)\quad\text{weakly star}.\tag{29}$$

Eq. (26) imply that

$$E_t^l\to E_t\quad\text{in}\quad L^\infty(\mathbb{R}^+,H^{-2})\quad\text{weakly star},\tag{30}$$

$$n_t^l \to n_t \quad \text{in} \quad L^\infty(\mathbb{R}^+, H^{-1}) \quad \text{weakly star,}$$

$$\varphi_t^l \to \varphi_t \quad \text{in} \quad L^\infty(\mathbb{R}^+, H^{-1}) \quad \text{weakly star.}$$

Moreover, let us note that the following maps are continuous.

$$H^1(\mathbb{R}^2) \to L^4(\mathbb{R}^2), \quad u \mapsto u,$$

$$H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad (u,v) \mapsto uv.$$

It then follows from eq. (27) and (28) that

$$\left|E^l\right|^2 \to w \quad \text{in} \quad L^\infty(\mathbb{R}^+, L^2) \quad \text{weakly star,} \tag{31}$$

$$n^l E^l \to z \quad \text{in} \quad L^\infty(\mathbb{R}^+, L^2) \quad \text{weakly star.} \tag{32}$$

First, we prove $w = |E|^2$. Let Ω be any bounded subdomain of $\mathbb{R}$. We notice that

the embedding $H^2(\Omega) \to L^4(\Omega)$ is compact,

and for any Banach space X,

the embedding $L^\infty(\mathbb{R}^+, X) \to L^2(0,T;X)$ is continuous.

Hence, according to eq.(27), (31) and Lemma 6, applied to $B_0 = H^2(\Omega),\ B = L^4(\Omega),\ B_1 = H^{-2}(\Omega)$, and says that some subsequence of $E^l|_\Omega$ (also labeled by l) converges strongly to $E|_\Omega$ in $L^2(0,T;L^4(\Omega))$. So we can assume that

$$E^l \to E \quad \text{strongly in} \quad L^2(0,T;L^4_{loc}(\Omega)), \tag{33}$$

and thus

$$E^l \to E \quad \text{a.e. in} \quad [0,T]\times\Omega.$$

Then, using Lemma 7 and eq. (31) imply that $w = |E|^2$.

Second, we prove $z = nE$. Let ψ be some test function in $L^2(0,T;H^1)$, $\operatorname{supp}\psi \subset \Omega \subset \mathbb{R}$.

$$\int_0^T\int_{\mathbb{R}} (n^l E^l - nE)\psi\, dxdt = \int_0^T\int_\Omega n^l\left(E^l - E\right)\psi\, dxdt + \int_0^T\int_\Omega \left(n^l - n\right)E\psi\, dxdt.$$

On one hand

$$\left|\int_0^T\int_\Omega n^l\left(E^l - E\right)\psi\, dxdt\right| \le \left\|n^l\right\|_{L^\infty(0,T;L^2(\Omega))}\left\|E^l - E\right\|_{L^2(0,T;L^4(\Omega))}\|\ \psi\|_{L^2(0,T;L^4(\Omega))}.$$

Since Ω is bounded, we deduce from eq. (28) and (33) that

$$\int_0^T\int_\Omega n^l\left(E^l - E\right)\psi\, dxdt \to 0 \quad (l \to +\infty).$$

On the other hand, let us note that $E\psi \in L^1(0,T;L^2)$. In fact

$$\|E\psi\|_{L^1(0,T;L^2)} \le \|E\|_{L^2(0,T;L^4)} \| \psi\|_{L^2(0,T;L^4)} < \infty.$$

Therefore we deduce from eq. (28) that

$$\int_0^T \int_\Omega (n^l - n)E\psi\, dxdt \to 0 \quad (l \to +\infty).$$

Thus $n^l E^l \to nE$ in $L^2(0,T;H^{-1})$. So $z = nE$.

Hence taking $l \to \infty$ from eq. (21)-(24), by using the density of ω_j in $H_0^2(\Omega)$ we get the existence of local generalized solution for the periodic initial value problem (4)-(7). letting $D \to \infty$, the existence of local solution for the initial value problem (4)-(7) can be obtain. By the continuation extension principle and a prior estimates, we can get the existence of global generalized solution for problem (4)-(7).

We thus complete the proof of Theorem 2. Hence one can get Theorem 1.

Conclusion

This paper considers the existence of the generalized solution to the initial vale problem for a generalized Zakharov equation by a priori integral estimates and Galerkin method, one has the existence of the global generalized solution to the problem.

Discussion

One can regard (1)-(2) as the Langmuir turbulence parameterized by H $(0 < H < 1)$ and study the asymptotic behavior of the systems (1)-(2) when H goes to zero.

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