AMSE JOURNALS-AMSE IIETA publication-2017-Series: Advances A; Vol. 54; N°3; pp 424–434
Submitted Oct. 25, 2017; Revised Dec. 26, 2017; Accepted Jan. 12, 2018
https://doi.org/10.18280/ama_a.540303

Some aspects of Equivalence Picture Fuzzy Relation

Palash Dutta, Kuntal Saikia

Dept. of Mathematics, Dibrugarh University-786001, India
(palash.dtt@gmail.com, nirvan.xperia@gmail.com)

Abstract

Most commonly uncertainty occurs because of vagueness, imprecision, partial information etc. To deal with this type of uncertainty, initially fuzzy set theory (FST) was explored and later, interval valued fuzzy set (IVFS) and intuitionistic fuzzy set (IFS) were developed and successfully applied in different areas. Although, IFS ascribes a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed one, but one of the important and integral part i.e., degree of neutrality is not taken into consideration in IFS, which is generally occurred. In such circumstances, picture fuzzy set (PFS) can be considered as a strong mathematical tool, which adequate in situations when human opinions involved more answers of type: yes, abstain, no. In this paper, an attempt has been made to study equivalence picture fuzzy relation and its some properties.

Key words

Fuzzy set, Picture fuzzy set, Picture fuzzy relations.

1. Introduction

After the developments of fuzzy set theory by Zadeh [1] various direct/indirect extensions of fuzzy set have been made and successfully applied in most of the problems of real world situation. an important generalization of fuzzy set theory is the theory of intuitionistic fuzzy set (IFS), introduced by Atanassov [2] ascribing a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed one. It is observed that fuzzy sets are IFSs but converse is not necessarily correct. Later IFS has been applied in different areas by various researchers. It is seen that one of the important concept of neutrality degree is lacking in IFS theory. Concept of neutrality degree can be seen in situations when we face human opinions involving more answers of type: yes, abstain, no, refusal. For example, in a democratic election station, the council issues 500 voting papers for a candidate. The voting results are divided into four groups accompanied with the number of papers namely "vote for" (300), "abstain" (64), "vote against" (115) and "refusal of voting" (21). Group "abstain" means that the voting paper is a whitepaper rejecting both "agree" and "disagree" for the candidate but still takes the vote. Group "refusal of voting" is either invalid voting papers or bypassing the vote. On the other hand, in medical diagnosis degree of neutrality can be considered. E.g., there may not have effect of the symptoms temperature, headache on the diseases stomach and chest problems. Similarly, the symptoms stomach pain and chest pain have neutral effect on the diseases viral fever, malaria, typhoid etc. In this regard, Cuong and Kreinovich [3] introduced Picture fuzzy set (PFS) which is a direct extension of fuzzy set and Intuitionistic fuzzy set by incorporating the concept of positive, negative and neutral membership degree of an element. Cuong [4] studied some properties of PFSs and suggested distance measures between PFSs. Phong et al., [5] studied some compositions of picture fuzzy relations, Cuong and Hai [6] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets and also constructed main operations for fuzzy inference processes in picture fuzzy systems. Cuong et al., [7] properties of an involutive picture negator and some corresponding De Morgan fuzzy triples on picture fuzzy sets, Son [8] proposed a new distance measure between PFSs and applied in fuzzy clustering, Cuong et al., [9] investigate the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets.

Relations are a suitable tool for describing correspondences between objects. In this paper, an attempt has been made to study equivalence picture fuzzy relation as well as its some properties such as equivalence class, intersection and union of equivalence relations. Some extensions of Basnet [10] will also be made.

2. Preliminaries:

In this section some basic concept of picture fuzzy set has been reviewed.

Definition 2.1 A Picture Fuzzy Set (PFS) **A** on a universe **X** is an object of the form

$$A=\left\{\left(x,\mu_A(x),\eta_A(x),\nu_A(x)\right)\mid x\in X\right\}$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A, $\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A.

$\mu_A(x), \eta_A(x), \nu_A(x)$ must satisfy the condition $\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1 \ \forall \ x \in X$.

Then $\forall \ x \in X$, $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called the degree of refusal membership of "x" in **A**.

Definition 2.2 If $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \eta_B(x), \nu_B(x)) \mid x \in X\}$ be any two PFS of a set **X** then

(1) $A \subseteq B$ iff $\forall \ x \in X$, $\mu_A(x) \le \mu_B(x), \eta_A(x) \ge \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.

(2) $A = B$ iff $\forall \ x \in X$, $\mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$.

(3) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x) = \nu_B(x))) \mid x \in X\}$

(4) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x) = \nu_B(x))) \mid x \in X\}$.

3. Picture Fuzzy Relation:

In this section, (α, β, γ)-cut of picture fuzzy relation has been defined. Reflexivity, symmetricity and transitivity of picture fuzzy relation are defined. Equivalence of picture fuzzy relation has been shown based on (α, β, γ)-cut. Later on, studied on equivalence class, intersection and union of equivalence picture fuzzy relations.

Definition 3.1 Let, A be a non-empty set. A picture fuzzy relation (PF relation) **R** on **A** is a picture fuzzy set

$$R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in A \times A\}$$

where $\mu_A(x) : A \times A \to [0,1]$, $\eta_A(x) : A \times A \to [0,1]$ and $\nu_A(x) : A \times A \to [0,1]$ satisfying the condition $\mu_A(x,y) + \eta_A(x,y) + \nu_A(x,y) \le 1 \ \forall \ (x,y) \in A \times A$.

Definition 3.2 A PF Relation $R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in A \times A\}$ is said to be reflexive $\mu_A(x,x) = 1$, $\eta_A(x,x) = 0$ and $\nu_A(x,x) = 0 \ \forall \ x \in A$. Also **R** is symmetric if $\mu_A(x,y) = \mu_A(y,x)$, $\eta_A(x,y) = \eta_A(y,x)$ and $\nu_A(x,y) = \nu_A(y,x), \forall (x,y) \in A \times A$.

Definition 3.3 If $R_1 = \{((x,y), \mu_1(x,y), \eta_1(x,y), \nu_1(x,y)) \mid (x,y) \in A \times A\}$ and $R_2 = \{((x,y), \mu_2(x,y), \eta_2(x,y), \nu_2(x,y)) \mid (x,y) \in A \times A\}$ be two PF Relations on **A** then **J** composition denoted by $R_1 \circ R_2$ is defined by

$$R_1 \circ R_2 = \{((x,y), \mu_1 \circ \mu_2(x,y), \eta_1 \circ \eta_2(x,y), \nu_1 \circ \nu_2(x,y)) \mid (x,y) \in A \times A\}$$

where $\mu_1 \circ \mu_2(x,y) = \sup\{\min\{\mu_1(x,z), \mu_2(z,y)\}\}$, $z \in A$

$$\eta_1 \circ \eta_2(x,y) = \inf\{\max\{\eta_1(x,z), \eta_2(z,y)\}\}, \ z \in A$$

$$\nu_1 \circ \nu_2(x,y) = \inf\{\max\{\nu_1(x,z), \nu_2(z,y)\}\}, \ z \in A.$$

Definition 3.4 (α, β, γ)-cut of a picture fuzzy relation **R**
$C_{\alpha,\beta,\gamma}(R) = \{(x,y) \in X \times X \mid \mu_R(x,y) \geq \alpha, \eta_R(x,y) \leq \beta, \nu_R(x,y) \leq \gamma\}$

Definition 3.5 A PF Relation **R** on **A** is called transitive if $R \circ R \subseteq R$.

Definition 3.6 A PF Relation **R** on **A** is called Picture Fuzzy Equivalence Relation if **R** is reflexive, symmetric and transitive.

Definition 3.7 For any PFS $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ of set **X**, we define a (α, β, γ) - cut of **A** as the crisp set $\{x \in X \mid \mu_A(x) \geq \alpha, \eta_A(x) \leq \beta, \nu_A(x) \leq \gamma\}$ of **X** and it is denoted by $C_{\alpha,\beta,\gamma}(A)$.

Theorem 3.1 Let, $R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ be a relation on a set **X**. Then **A** is a PF equivalence on **X** iff $C_{\alpha,\beta,\gamma}(R)$ is an equivalence relation on **X**, with $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma \leq 1$.

Proof. We have,

$$C_{\alpha,\beta,\gamma}(R) = \{(x,y) \in X \times X \mid \mu_R(x,y) \geq \alpha, \eta_R(x,y) \leq \beta, \nu_R(x,y) \leq \gamma\}$$

$\because$ **R** is a PF equivalence relation so

$\mu_A(x,x) = 1 \geq \alpha$, $\eta_A(x,x) = 0 \leq \beta$ and $\nu_A(x,x) = 0 \leq \lambda \ \forall \ x \in X$.

$\therefore \quad (x,x) \in C_{\alpha,\beta,\gamma}(R)$

$\Rightarrow \quad C_{\alpha,\beta,\gamma}(R)$ is reflexive.

Now,

Let, $(x,y) \in C_{\alpha,\beta,\gamma}(R)$

Then $\mu_R(x,y) \geq \alpha, \eta_R(x,y) \leq \beta$ and $\nu_R(x,y) \leq \gamma$

But, **R** is PF equivalence so

$$\mu_R(y,x) = \mu_R(x,y) \geq \alpha$$

$$\eta_R(y,x) = \eta_R(x,y) \leq \beta$$

$\nu_R(y,x)=\nu_R(x,y)\leq\gamma$.

$\therefore\quad (y,x)\in C_{\alpha,\beta,\gamma}(R)$

$\Rightarrow\quad C_{\alpha,\beta,\gamma}(R)$ is symmetric.

Again,

Let, $(x,y)\in C_{\alpha,\beta,\gamma}(R)$ and $(y,z)\in C_{\alpha,\beta,\gamma}(R)$

Then $\mu_R(x,y)\geq\alpha, \eta_R(x,y)\leq\beta$ and $\nu_R(x,y)\leq\gamma$

and $\mu_R(y,\mathrm{z})\geq\alpha, \eta_R(y,z)\leq\beta$ and $\nu_R(y,z)\leq\gamma$

$\therefore\quad \min\{\mu_R(x,y),\mu_R(y,z)\}\geq\alpha$

$\max\{\eta_R(x,y),\eta_R(y,z)\}\leq\beta$

and $\quad \max\{\nu_R(x,y),\nu_R(y,z)\}\leq\gamma$.

$\Rightarrow \max\{\min\{\mu_R(x,y),\mu_R(y,z)\}\}\geq\alpha \;\Rightarrow(\mu_R\circ\mu_R)(x,z)\geq\alpha$

$\min\{\max\{\eta_R(x,y),\eta_R(y,z)\}\}\leq\beta \;\Rightarrow(\eta_R\circ\eta_R)(x,z)\leq\beta$

and $\min\{\max\{\nu_R(x,y),\nu_R(y,z)\}\}\leq\gamma \;\Rightarrow(\nu_R\circ\nu_R)(x,z)\leq\gamma$

But, **R** is PF equivalence so

$\mu_R(x,z)\geq(\mu_R\circ\mu_R)(x,z)\geq\alpha$

$\eta_R(x,z)\leq(\eta_R\circ\eta_R)(x,z)\leq\beta$

and $\quad \nu_R(x,z)\leq(\nu_R\circ\nu_R)(x,z)\leq\gamma$.

$\therefore\quad (x,\mathrm{z})\in C_{\alpha,\beta,\gamma}(R)$

$\Rightarrow\quad C_{\alpha,\beta,\gamma}(R)$ is transitive.

Conversely,

Suppose that $C_{\alpha,\beta,\gamma}(R)$ is an equivalence relation on **X**

Taking $\alpha=1,\ \beta=0$ and $\gamma=0$ we get $C_{1,0,0}(R)$ is equivalence and so a reflexive relation and so $(x,x)\in C_{1,0,0}(R)\ \ \forall\ \ x\in X$.

$\therefore\quad \mu_R(x,x)\geq 1,\ \eta_R(x,x)\leq 0$ and $\nu_R(x,x)\leq 0$

Thus, $\mu_R(x,x)=1,\ \eta_R(x,x)=0$ and $\nu_R(x,x)=0$.

$\therefore$ PF relation **R** is reflexive.

For any $x, y \in X$, let $\mu_R(x,y)=\alpha$, $\eta_R(x,y)=\beta$ and $\nu_R(x,y)=\gamma$.

Then $\alpha+\beta+\gamma \le 1$ and so by hypothesis $C_{\alpha,\beta,\gamma}(R)$ is equivalence and hence symmetric relation on **X**. Also, $(x,y)\in C_{\alpha,\beta,\gamma}(R)$ so by symmetry $(y,x)\in C_{\alpha,\beta,\gamma}(R)$.

$\therefore$ $\mu_R(y,x)\ge\alpha=\mu_R(x,y)$, $\eta_R(y,x)\le\beta=\eta_R(x,y)$ and $\nu_R(y,x)\le\gamma=\nu_R(x,y)$.

Similarly, if $\mu_R(y,x)=\delta$, $\eta_R(y,x)=\sigma$ and $\nu_R(y,x)=\phi$ then $(x,y)\in C_{\delta,\sigma,\phi}(R)$

Now, $\mu_R(x,y)\ge\delta=\mu_R(y,x)$, $\eta_R(x,y)\le\sigma=\eta_R(y,x)$ and $\nu_R(x,y)\le\phi=\nu_R(y,x)$.

Hence, $\mu_R(x,y)=\mu_R(y,x)$, $\eta_R(x,y)=\eta_R(y,x)$ and $\nu_R(x,y)=\nu_R(y,x)$.

$\therefore$ PF relation **R** is symmetric.

Again,

Let, $x, y, z \in X$

$$\min\{\mu_R(x,z),\mu_R(z,y)\}=\alpha$$

$$\max\{\eta_R(x,z),\eta_R(z,y)\}=\beta$$

and $$\max\{\nu_R(x,z),\nu_R(z,y)\}=\gamma .$$

Then $\alpha\ge 0$, $\beta<1$, $\gamma<1$ and $\alpha+\beta+\gamma\le 1$.

$\therefore$ $C_{\alpha,\beta,\gamma}(R)$ is an equivalence relation on **X**.

$\because$ $\mu_R(x,z)\ge\alpha$, $\mu_R(z,y)\ge\alpha$, $\eta_R(x,z)\le\beta$, $\eta_R(z,y)\le\beta$ and $\nu_R(x,z)\le\gamma$, $\nu_R(z,y)\le\gamma$.

So, $(x,z)\in C_{\alpha,\beta,\gamma}(R)$ and $(z,y)\in C_{\alpha,\beta,\gamma}(R)$.

As, $C_{\alpha,\beta,\gamma}(R)$ is equivalence relation so by transitivity $(x,y)\in C_{\alpha,\beta,\gamma}(R)$.

Then $\mu_R(x,y)\ge\alpha,\eta_R(x,y)\le\beta$ and $\nu_R(x,y)\le\gamma$.

$$\Rightarrow \quad \mu_R(x,y)\ge\alpha=\min\{\mu_R(x,z),\mu_R(z,y)\}$$

$$\eta_R(x,y)\le\beta=\max\{\eta_R(x,z),\eta_R(z,y)\}$$

and $$\nu_R(x,y)\le\gamma=\max\{\nu_R(x,z),\nu_R(z,y)\} .$$

$$\Rightarrow \quad \mu_R(x,y)\ge\sup\{\min\{\mu_R(x,z),\mu_R(z,y)\}\}$$

$$\eta_R(x,y) \le \inf\{\max\{\eta_R(x,z), \eta_R(z,y)\}\}$$

and $$\nu_R(x,y) \le \inf\{\max\{\nu_R(x,z), \nu_R(z,y)\}\}.$$

$\Rightarrow$ $$\mu_R(x,y) \ge (\mu_R \circ \mu_R)(x,y)$$

$$\eta_R(x,y) \le (\eta_R \circ \eta_R)(x,y)$$

and $$\nu_R(x,y) \le (\nu_R \circ \nu_R)(x,y).$$

$\Rightarrow$ $\mu_R \supseteq \mu_R \circ \mu_R$, $\eta_R \subseteq \eta_R \circ \eta_R$ and $\nu_R \subseteq \nu_R \circ \nu_R$.

$\Rightarrow$ PF relation **R** is transitive.

Hence, PF relation is an equivalence relation.

Definition 3.7 Let $R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ be a PF equivalence on a set **X**. Let, **a** be an element of **X**. Then the PFS defined by $aR = \{(x, a\mu_R(x), a\eta_R(x), a\nu_R(x) \mid x \in X)\}$ where $(a\mu_R)(x) = \mu_R(a,x)$, $(a\eta_R)(x) = \eta_R(a,x)$ and $(a\nu_R)(x) = \nu_R(a,x)$ $\forall$ $x \in X$ is called and PF equivalence class of **a** with respect to **R**.

Theorem 3.2 Let $R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ be a PF equivalence relation on a set **X**. Let, **a** be any element of **X**. Then for $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma \le 1$, $C_{\alpha,\beta,\gamma}(aR) = [a]$, the equivalence class of **a** with the equivalence relation $C_{\alpha,\beta,\gamma}(R)$ in **X**.

Proof: We have,

$$[a] = \{x \in X \mid (a,x) \in C_{\alpha,\beta,\gamma}(R)\}$$

$$= \{x \in X \mid \mu_R(a,x) \ge \alpha, \eta_R(a,x) \le \beta, \nu_R(a,x) \le \gamma\}$$

$$= \{x \in X \mid (a\mu_R)(x) \ge \alpha, (a\eta_R)(x) \le \beta, (a\nu_R)(x) \le \gamma\}$$

$$= C_{\alpha,\beta,\gamma}(aR).$$

Theorem 3.3 Let $R = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ be a PF equivalence relation on a set **X**. Then $[a] = [b]$ iff $(a,b) \in C_{\alpha,\beta,\gamma}(R)$ where $[a],[b]$ are equivalence classes of **a** and **b** with respect to the equivalence relation $C_{\alpha,\beta,\gamma}(R)$ in **X** for $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma \le 1$.

Proof. Let, $[a] = [b]$

Then $$C_{\alpha,\beta,\gamma}(aR) = C_{\alpha,\beta,\gamma}(bR)$$

$\Rightarrow$ $\{x \in X \mid (a\mu_R)(x) \geq \alpha, (a\eta_R)(x) \leq \beta, (a\nu_R)(x) \leq \gamma\} =$

$\{x \in X \mid (b\mu_R)(x) \geq \alpha, (b\eta_R)(x) \leq \beta, (b\nu_R)(x) \leq \gamma\}$

Let, $x \in C_{\alpha,\beta,\gamma}(aR) = C_{\alpha,\beta,\gamma}(bR)$

$\Rightarrow$ $(a\mu_R)(x) \geq \alpha, (a\eta_R)(x) \leq \beta, (a\nu_R)(x) \leq \gamma$ and
$(b\mu_R)(x) \geq \alpha, (b\eta_R)(x) \leq \beta, (b\nu_R)(x) \leq \gamma$

$\Rightarrow$ $(\mu_R)(a,x) \geq \alpha, (\eta_R)(a,x) \leq \beta, (\nu_R)(a,x) \leq \gamma$ and
$(\mu_R)(b,x) \geq \alpha, (\eta_R)(b,x) \leq \beta, (\nu_R)(b,x) \leq \gamma$

$\Rightarrow$ $\min\{\mu_R(a,x), \mu_R(b,x)\} \geq \alpha$, $\max\{\eta_R(a,x), \eta_R(b,x)\} \leq \beta$ and
$\max\{\nu_R(a,x), \nu_R(b,x)\} \leq \gamma$

$\Rightarrow$ $\sup\{\min\{\mu_R(a,x), \mu_R(b,x)\}\} \geq \alpha$, $\inf\{\max\{\eta_R(a,x), \eta_R(b,x)\}\} \leq \beta$
and $\inf\{\max\{\nu_R(a,x), \nu_R(b,x)\}\} \leq \gamma$

$\Rightarrow$ $(\mu_R \circ \mu_R)(a,b) \geq \alpha$, $(\eta_R \circ \eta_R)(a,b) \leq \beta$ and $(\nu_R \circ \nu_R)(a,b) \leq \gamma$.

$\Rightarrow$ $(a,b) \in C_{\alpha,\beta,\gamma}(R)$

Conversely,

Let, $(a,b) \in C_{\alpha,\beta,\gamma}(aR)$

$\Rightarrow$ $\mu_R(a,b) \geq \alpha$, $\eta_R(a,b) \leq \beta$, $\nu_R(a,b) \leq \gamma$ $\cdots\cdots(1)$

Let, $x \in C_{\alpha,\beta,\gamma}(R)$

Then $(a\mu_R)(x) \geq \alpha, (a\eta_R)(x) \leq \beta, (a\nu_R)(x) \leq \gamma$

$\Rightarrow$ $(\mu_R)(a,x) \geq \alpha, (\eta_R)(a,x) \leq \beta, (\nu_R)(a,x) \leq \gamma$

$\Rightarrow$ $\min\{\mu_R(b,a), \mu_R(a,x)\} \geq \alpha$, $\max\{\eta_R(b,a), \eta_R(a,x)\} \leq \beta$ and
$\max\{\nu_R(b,a), \nu_R(a,x)\} \leq \gamma$ (using (1))

$\Rightarrow$ $\sup\{\min\{\mu_R(b,a), \mu_R(a,x)\}\} \geq \alpha$, $\inf\{\max\{\eta_R(b,a), \eta_R(a,x)\}\} \leq \beta$
and $\inf\{\max\{\nu_R(b,a), \nu_R(a,x)\}\} \leq \gamma$

$\Rightarrow$ $(\mu_R \circ \mu_R)(b,x) \geq \alpha$, $(\eta_R \circ \eta_R)(b,x) \leq \beta$ and $(\nu_R \circ \nu_R)(b,x) \leq \gamma$

$\Rightarrow$ $(b\mu_R)(x) \geq \alpha, (b\eta_R)(x) \leq \beta$ and $(b\nu_R)(x) \leq \gamma$

$$\Rightarrow \quad (a,b) \in C_{\alpha,\beta,\gamma}(bR)$$

$$\Rightarrow \quad C_{\alpha,\beta,\gamma}(aR) \subseteq C_{\alpha,\beta,\gamma}(bR)$$

Similarly, $C_{\alpha,\beta,\gamma}(bR) \subseteq C_{\alpha,\beta,\gamma}(aR)$

Hence, $C_{\alpha,\beta,\gamma}(aR) = C_{\alpha,\beta,\gamma}(bR)$

$$\Rightarrow \quad [a] = [b].$$

Theorem 3.4 The intersection of two PF equivalence relations on a set is again a PF equivalence relation on the set.

Proof. Let, $A = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ and

$B = \{((x,y), \mu_B(x,y), \eta_B(x,y), \nu_B(x,y)) \mid (x,y) \in X \times X\}$ be two PF equivalence relations on a set **X**.

For any $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma \le 1$, we have $C_{\alpha,\beta,\gamma}(A \cap B) = C_{\alpha,\beta,\gamma}(A) \cap C_{\alpha,\beta,\gamma}(B)$.

We know that, $C_{\alpha,\beta,\gamma}(A)$ and $C_{\alpha,\beta,\gamma}(B)$ are equivalence relations on **X** and so being intersection of two equivalence relations $C_{\alpha,\beta,\gamma}(A \cap B)$ is also an equivalence relation on **X** and $A \cap B$ is an PF relation on **X**.

Remark 3.5: Union of two PF equivalence relations on a set is not necessarily a PF equivalence relation on the set.

Let, $X = \{a,b,c\}$.

$A = \{((x,y), \mu_A(x,y), \eta_A(x,y), \nu_A(x,y)) \mid (x,y) \in X \times X\}$ and

$B = \{((x,y), \mu_B(x,y), \eta_B(x,y), \nu_B(x,y)) \mid (x,y) \in X \times X\}$ be two PFS on X where $\mu_A(a,a) = \mu_A(b,b) = \mu_A(c,c) = 1$, $\mu_A(a,b) = \mu_A(b,a) = \mu_A(a,c) = \mu_A(a,c) = 0.2$ and $\mu_A(b,\mathrm{c}) = \mu_A(c,b) = 0.7$.

$\eta_A(a,a) = \eta_A(b,b) = \eta_A(c,c) = 0$, $\eta_A(a,b) = \eta_A(b,a) = \eta_A(a,c) = \eta_A(a,c) = 0.5$ and $\eta_A(b,\mathrm{c}) = \eta_A(c,b) = 0.2$

$\nu_A(a,a) = \nu_A(b,b) = \nu_A(c,c) = 0$, $\nu_A(a,b) = \nu_A(b,a) = \nu_A(a,c) = \nu_A(c,a) = 0.3$ and $\nu_A(b,c) = \nu_A(c,b) = 0.1$

Again,

$\mu_B(a,a) = \mu_B(b,b) = \mu_B(c,c) = 1$, $\mu_B(a,b) = \mu_B(b,a) = \mu_B(b,c) = \mu_B(c,b) = 0.4$ and $\mu_B(a,\mathrm{c}) = \mu_B(c,a) = 0.6$

$\eta_B(a,a)=\eta_B(b,b)=\eta_B(c,c)=0$, $\eta_B(a,b)=\eta_B(b,a)=\eta_B(b,c)=\eta_B(c,b)=0.3$ and $\eta_B(a,c)=\eta_B(c,a)=0.2$

and $\nu_B(a,a)=\nu_B(b,b)=\nu_B(c,c)=0$, $\nu_B(a,b)=\nu_B(b,a)=\nu_B(b,c)=\nu_B(c,b)=0.3$ and $\nu_B(b,c)=\nu_B(c,b)=0.1$

Now,

$$A\cup B=\left\{\left((x,y),(\mu_A,\mu_B),(\eta_A,\eta_B),(\nu_A,\nu_B)\right)\mid(x,y)\in X\times Y\right\}$$

It is not transitive as

$$\left\{(\mu_A\cup\mu_B)\circ(\mu_A\cup\mu_B)\right\}(a,b)$$

$$=\sup\left\{\begin{array}{l}\min\left\{(\mu_A\cup\mu_B)(a,a),(\mu_A\cup\mu_B)(a,b)\right\},\min\left\{(\mu_A\cup\mu_B)(a,b),(\mu_A\cup\mu_B)(b,b)\right\},\\ \min\left\{(\mu_A\cup\mu_B)(a,c),(\mu_A\cup\mu_B)(c,\mathrm{b})\right\}\end{array}\right\}$$

$$=\sup\left\{\min\{1,0.4\},\min\{0.4,1\},\min\{0.7,0.6\}\right\}$$

$$=0.6\geq 0.4=\max\left\{\mu_A(a,b),\mu_B(a,b)\right\}=\mu_A\cup\mu_B(a,b).$$

$\therefore$ $A\cup B$ is not an PF equivalence relation on X.

4. Conclusion

Reflexivity, symmetricity and transitivity of a picture fuzzy relation have been defined. Equivalence of picture fuzzy relation has been proved based on (α,β,γ)-cut. Some properties on equivalence class are discussed. Intersection and union of equivalence PF relation are also studied. It is found that intersection of equivalence PF relation is also equivalence PF relation, whereas union is not.

References

1. L.A. Zadeh, Fuzzy sets, 1965, Inform. Control, vol. 8, pp. 338–356.
2. K.T. Atanassov, Intuitionistic fuzzy sets, 1986, Fuzzy Sets and Systems, vol. 20, pp. 87–96.
3. B. C. Cuong, V. Kreinovich, Picture Fuzzy Sets- a new concept for computational intelligence problems, 2013, Proceedings of the Third World Congress on Information and Communication Technologies WIICT 2013, pp. 1-6.
4. B. C. Cuong, Picture Fuzzy Sets, 2014, Journal of Computer Science and Cybernetics, vol. 30, no. 4, pp. 409–420.

5. P. H. Phong, D. T. Hieu, R. H. Ngan, P. T. Them, (2014), Some Compositions of Picture Fuzzy Relations, *Kỷ yếu Hội nghị Quốc gia lần thứ … về Nghiên cứu cơ bản và ứng dụng Công Nghệ thông tin (FAIR); …, ngày …/…/…*

6. B. C. Cuong, P. V. Hai, Some fuzzy logic operators for picture fuzzy sets, 2015, Seventh International Conference on Knowledge and Systems Engineering, pp. 132-137.

7. B. C. Cuong, R. T. Ngan, B. D. Hai, An involutive picture fuzzy negator on picture fuzzy sets and some De Morgan triples, 2015, Seventh International Conference on Knowledge and Systems Engineering, 126-131.

8. L.H. Son, Generalized picture distance measure and applications to picture fuzzy clustering, 2016, Appl. Soft Comput. J.,

9. Bui Cong Cuong, Vladik Kreinovich, Roan Thi Ngan, A classification of representable t-norm operators for picture fuzzy sets, 2016, Departmental Technical Reports (CS), Paper 1047, http://digitalcommons.utep.edu/cs_techrep/1047.

10. D. K. Basnet, N. K. Sarma, A note on intuitionistic fuzzy equivalence relation, 2010, International Mathematical Forum, vol. 5, no. 67, pp. 3301-3307.